PHYS Solutions (2/22 assignment)

#21) $g = \frac{GM_e}{r^2}$ $M_e = 5.97 \times 10^{24}$ kg

$G = 6.67 \times 10^{-11} \; \text{J} \cdot \text{m/kg}^2$

$\underbrace{(\text{kg m}^2/\text{s}^2)}_{\text{J}} \text{m/kg}^2 = \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$

$r = R_e + h$ where $\begin{cases} R_e = 6.4 \times 10^3 \text{ km} = 6.4 \times 10^6 \text{ m} \\ h = 360 \text{ km to } 400 \text{ km} \end{cases}$

$r = 6.8 \times 10^6$ m (about $\underbrace{4 \times 10^5 \text{ m}}_{0.4 \times 10^6 \text{ m}}$)

(G, M_e, R_e in Appendix 5 (page 872))

$$g = 8.6 \text{ m/s}^2 \left(= \frac{[6.67 \times 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)][5.97 \times 10^{24} \text{ kg}]}{(6.8 \times 10^6)^2 \text{ m}^2} \right)$$

2 sig figs 3 sig figs 2 sig figs 3 sig figs

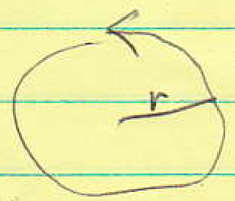
$2 = \min(3, 2, 3)$.

#22a) $v = \sqrt{gr}$ for circular orbits (p. 97)

$g = \frac{GM_e}{r^2} \rightarrow v = \sqrt{\frac{GM_e}{r}}$ $v = \frac{2\pi r}{T}$ ($= \frac{\text{circumference}}{\text{time}}$)

$\frac{2\pi r}{T} = \sqrt{\frac{GM_e}{r}}$

Because circular orbits have constant speed. (p. 96)



$4\pi^2 r^2 / T^2 = GM_e / r$

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM_e}{r^{2}} \rightarrow 4\pi^2 r^3 = GM_e T^2 \quad \cancel{\rightarrow 4\pi r^3}$$

$$\cancel{\frac{4\pi r^3}{GM_e} = T^2 \rightarrow \sqrt{\frac{4\pi r^3}{GM_e}} = T}$$

$$\rightarrow r^3 = \frac{GM_e T^2}{4\pi^2} \rightarrow r = \sqrt[3]{\frac{GM_e T^2}{4\pi}}$$

$$r = \sqrt[3]{\left(6.67\times10^{-11}\ \frac{m^3}{kg\cdot s^2}\right)\left(5.97\times10^{24}\,kg\right)\left(8.64\times10^4\right)^2 s^2 \Big/ \left(4\pi^2\right)}$$

$$T = 24\text{ hours}\left(\frac{60\text{ min}}{hr}\right)\left(\frac{60\text{ sec}}{min}\right) = 86400s$$

$$r = 4.22\times10^7\ m \qquad \text{altitude} = r - \underbrace{R_e}_{0.64\times10^7 m} = \boxed{3.58\times10^7 m}$$

distance to center of earth

#22b) $$g = \frac{GM_e}{r^2} = 3.10\times10^{-1}\ m/s^2$$

$$\frac{g_{sat}}{g_{ground}} = \frac{3.10\times10^{-1}}{9.8} = 3.2\times10^{-2} = \boxed{3.2\%}$$

~~#24)~~ $\cancel{T \propto r^{3/2}}$, ~~so the T-ratio is~~

$$\cancel{\left(\frac{30}{6}\right)^{3/2} = 5^{3/2} = 5\sqrt{5} \approx 11}$$

#24) $T \propto r^{3/2}$ (i.e., $T = kr^{3/2}$ for some constant k)

$T^{2/3} \propto (r^{3/2})^{2/3} = r$

$\Rightarrow r \propto T^{2/3} \Rightarrow$ r-ratio is $\left(\frac{30}{6}\right)^{2/3}$

$= \boxed{5^{2/3}} \approx 2.9$

#25a) See Example 16. (p. 100).

$E_i = K_i + U_i = \frac{1}{2}mv_i^2 - \frac{GMm}{r_i}$; $E_f = \frac{1}{2}mv_f^2 - \frac{GMm}{r_f}$

To get to arbitrarily large r_f,

$E_f \geq 0$ because $E_f \geq 0 - \frac{GMm}{r_f}$

and $\frac{GMm}{r_f}$ gets arbitrarily small as r gets large.

By conservation of energy, $E_i = E_f \geq 0$, so

$E_i \geq 0$, so $\frac{1}{2}mv_i^2 - \frac{GMm}{r_i} \geq 0$, so

$\frac{1}{2}\cancel{m}v_i^2 \geq \frac{GM\cancel{m}}{r_i}$, so $v_i \geq \sqrt{\frac{2GM}{r_i}}$.

So, escape velocity is $\boxed{v = \sqrt{\frac{2GM}{r}}}$. (not final answer)

$$\frac{dr}{dt} = \sqrt{\frac{2GM}{r}} \Rightarrow dr = \sqrt{\frac{2GM}{r}}\, dt$$

$$\Rightarrow \sqrt{\frac{r}{2GM}}\, dr = dt \Rightarrow \int \frac{r^{1/2}\, dr}{\sqrt{2GM}} = \int dt$$

$$\Rightarrow \frac{1}{\sqrt{2GM}} \frac{r^{3/2}}{3/2} = t + C$$

Choose a time coordinate system such that $C = 0$.

$$r^{3/2} = \frac{2}{3}\sqrt{2GM}\, t$$

$$r^3 = \frac{4}{9} 2GM t^2 \qquad \boxed{r = \sqrt[3]{\frac{8}{9} GM t^2}}$$

$$v = \frac{dr}{dt} = \frac{d\left(\sqrt[3]{8GM/9}\; t^{2/3}\right)}{dt}$$

$$= \sqrt[3]{8GM/9}\; t^{2/3-1} (2/3) = \boxed{\frac{2}{3}\sqrt[3]{8GM/9}\; t^{-1/3}}$$

$v(t)$

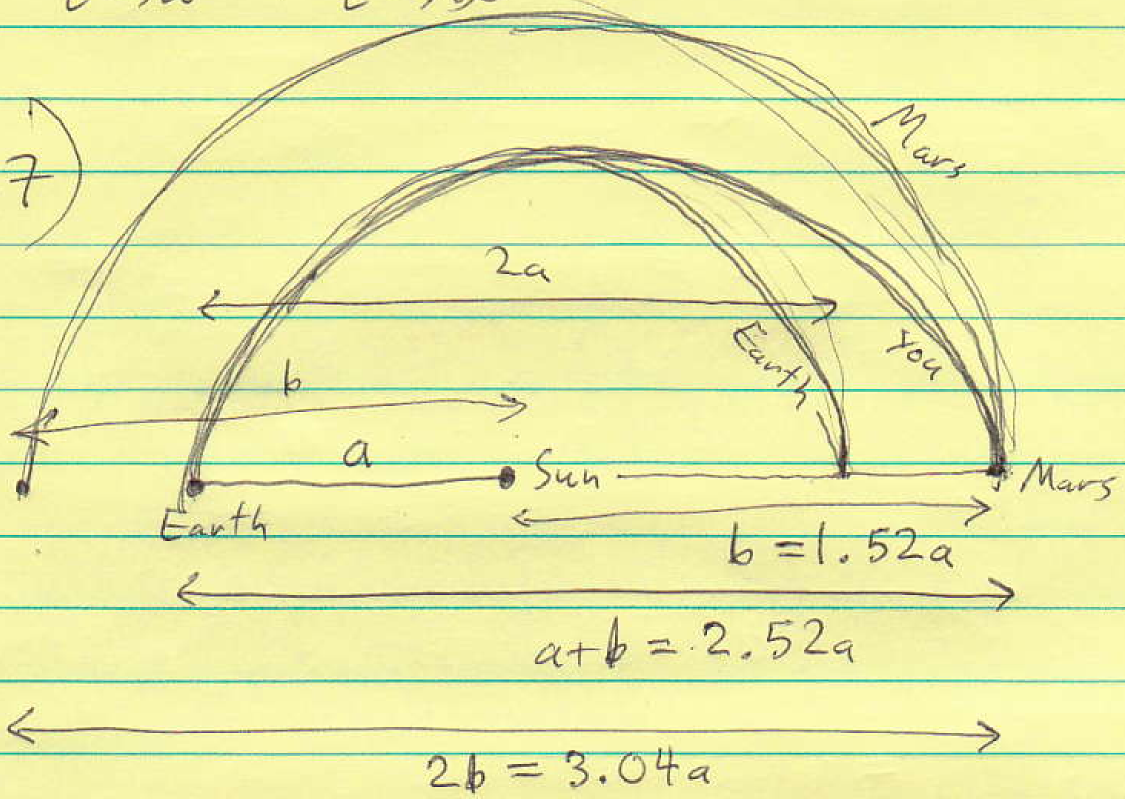
25b) $\sqrt[3]{\frac{8}{9}GMt^2}$: $\sqrt[3]{\left(\frac{m^3}{\cancel{kg}\cdot \cancel{s^2}}\right)(\cancel{kg})(\cancel{s^2})} = m$ ✓

$\frac{2}{3}\sqrt[3]{\frac{8}{9}GM} \cdot t^{-1/3}$: $\sqrt[3]{\left(\frac{m^3}{\cancel{kg}\cdot s^2}\right)(\cancel{kg})} \cdot s^{-1/3}$

$$= \frac{m}{s^{2/3}} \cdot \frac{1}{s^{1/3}} = \frac{m}{s}$$ ✓

25c) $\lim_{t \to \infty} r = \lim_{t \to \infty} \sqrt[3]{\frac{8}{9} GM}\, t^{2/3} = \infty$ ✓

#27)



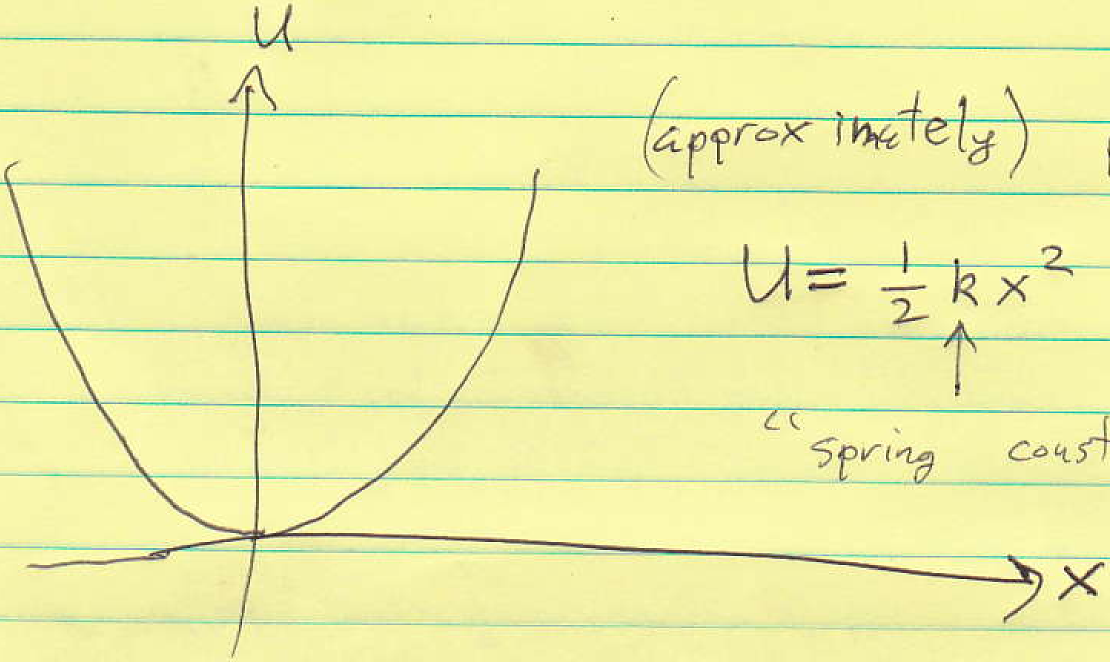
$T_{earth} = 1.00$ years; $T \propto \ell^{3/2}$

where ℓ = long axis length of ellipse.

$$\frac{T_{you}}{T_{earth}} = \left(\frac{2.52a}{2a}\right)^{3/2} = 1.41$$

$\Rightarrow$ your orbit has a period of 1.41 years

$\Rightarrow$ your trip lasts $\frac{1}{2}(1.41) =$ 0.705 years

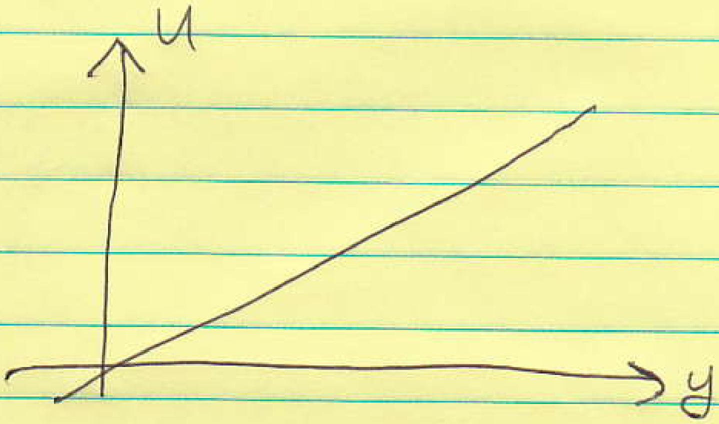


(approximately) parabolic

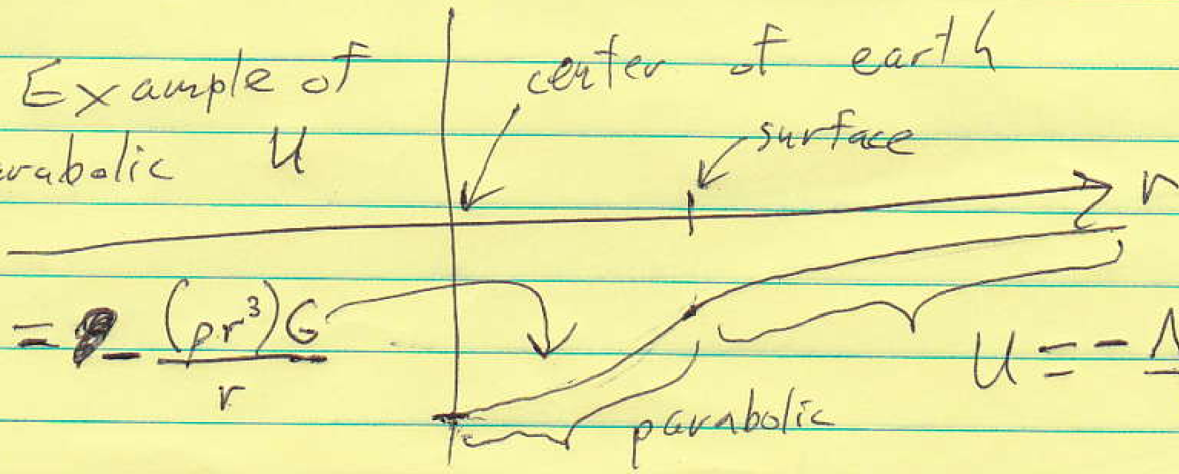
$$U = \frac{1}{2} k x^2$$

"spring constant"

$U = mgy$



Example of parabolic U



$$U = -\frac{(\rho r^3) G}{r}$$

$$U = -\frac{MG}{r}$$

constant $E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

oscillation

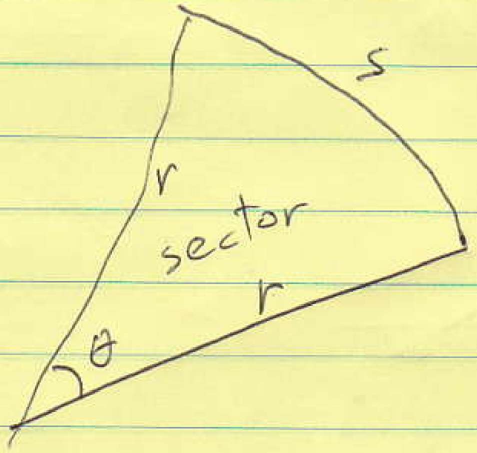
$$x = A\cos(\omega t + \varphi)$$

A, φ depend on

x, v at time 0.

$$v = \frac{dx}{dt} = -A\omega \sin(\overbrace{\omega t + \varphi}^{\text{radians}})$$

angular frequency



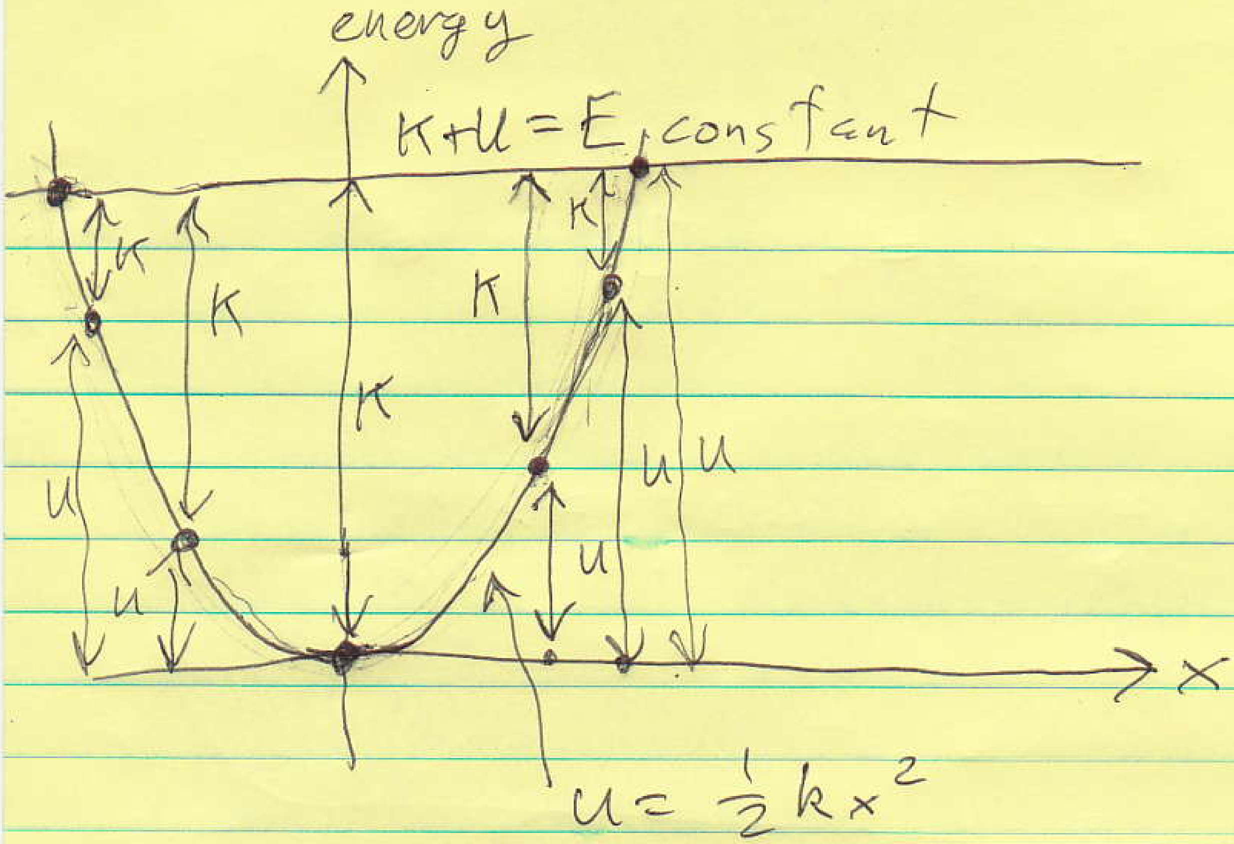
θ (radians) $= \frac{s}{r}$

θ has no units

1 radian = 1

$1^\circ = \frac{\pi}{180}$

units: ω: rad/s (same as 1/s)
t: s
φ: rad (same as no units)



$$x = A\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$K = \frac{1}{2} m v^2$$

$$U = \frac{1}{2} k x^2$$

constant

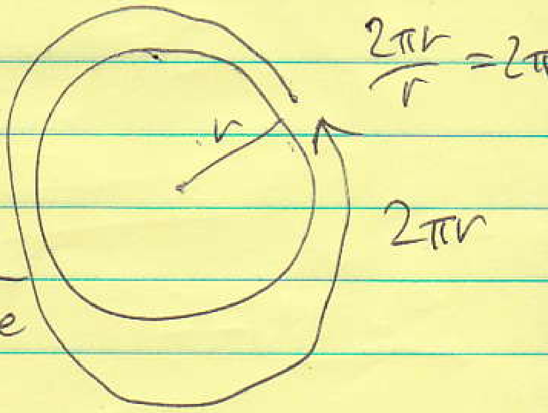
$$T = \text{period} = \frac{\text{time}}{\text{cycle}}$$

$$\omega = \frac{\text{radians}}{\text{time}}$$

1 cycle = 2π rad.

$$\frac{1}{\omega} = \frac{\text{time}}{\text{radian}}$$

$$T = \frac{1}{2\pi\omega} = \frac{\text{time}}{2\pi\ \text{rad}} = \frac{\text{time}}{\text{cycle}}$$



$$\omega = \sqrt{\frac{k}{m}} \Rightarrow T = \frac{1}{2\pi\sqrt{k/m}} = \frac{1}{2\pi}\sqrt{\frac{m}{k}}$$

$$f = \text{frequency} = \frac{\text{cycles}}{\text{time}} \qquad 1\ \text{Hz} = \frac{1}{s}$$

$$f = \frac{2\pi\ \text{rad}}{\text{time}} = 2\pi\omega$$

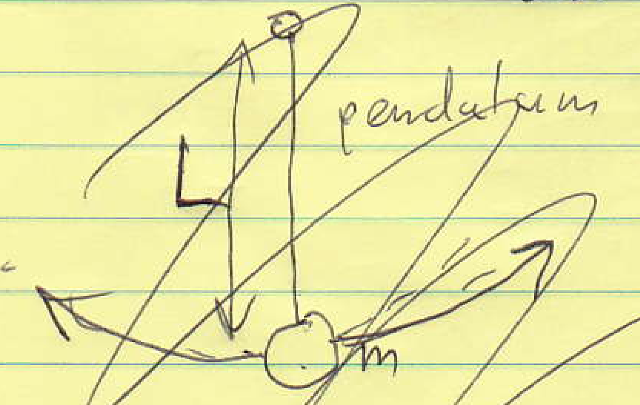
How do we find k?

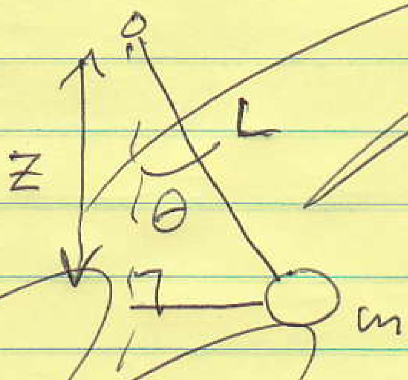
$$\frac{d^2U}{dx^2} \quad \text{at} \quad x = 0$$

$$U = \frac{1}{2}kx^2$$

$$\frac{dU}{dx} = \frac{1}{2}k(2x) = kx$$

$$\frac{d^2U}{dx^2} = k$$





$$z = L\cos\theta$$

$$U = -mgz$$

$$U = -mgL\cos\theta$$

$$\frac{dU}{d\theta} = mgL\sin\theta$$

$$\frac{d^2U}{d\theta^2} = mgL\cos\theta$$

Plug in $\theta = 0$

$$k = mgL(1)$$

$\cancel{\text{For small } \theta,\ U \approx \frac{1}{2} k \theta^2}$

$k = mgL$

$\cancel{T = \frac{1}{2\pi}\sqrt{\frac{\text{[illegible]}}{m}} = \frac{1}{2\pi}\sqrt{\frac{mgL}{m}} = \frac{\sqrt{gL}}{2\pi}}$

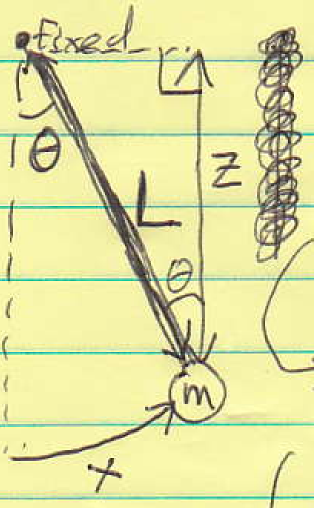
$\cancel{T = \frac{1}{2\pi}\sqrt{\frac{m}{k}} = \frac{1}{2\pi}\sqrt{\frac{m}{mgL}} = \frac{1}{2\pi\sqrt{gL}}}$

Due 3/1:

HW: #31, 32, 33, 34 (Ch. 2)

Reading: 3.1

Pendulum (corrected)



$$\theta = \frac{x}{L}$$

$$\cos\theta = \frac{z}{L}$$

$$z = L\cos\frac{x}{L}$$

$$U = -mgz = -mgL\cos\left(\frac{x}{L}\right)$$

$$\frac{dU}{dx} = mg\sin\left(\frac{x}{L}\right)$$

$$\frac{d^2U}{dx^2} = \frac{mg}{L}\cos\left(\frac{x}{L}\right)$$

$$k = \left(\frac{d^2U}{dx^2} \text{ at } x=0\right) = \frac{mg}{L}\cos\left(\frac{0}{L}\right) = \frac{mg}{L}$$

$$U \approx \frac{1}{2} k x^2 \quad \text{for } x \text{ small}$$

$$U \approx \frac{mg}{2L} x^2 \qquad T \approx \frac{1}{2\pi}\sqrt{\frac{m}{k}}$$

$$\Rightarrow T \approx \frac{1}{2\pi}\sqrt{\frac{m}{mg/L}} = \frac{1}{2\pi}\sqrt{\frac{1}{g/L}} = \frac{1}{2\pi}\sqrt{\frac{L}{g}}$$