Sequences

Examples:

$$\{1,-1,1,-1,1,-1,1,\dots\} = \underbrace{\{(-1)^n\}_{n=0}^{\infty}}_{\text{very precise and short to write}} = \{\underset{1}{(-1)^0}, \underset{-1}{(-1)^1}, \underset{1}{(-1)^2}, \underset{-1}{(-1)^3}, \underset{1}{(-1)^4}, \dots\}$$

$$\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty} = \left\{\frac{(-1)^1}{1}, \frac{(-1)^2}{2}, \frac{(-1)^3}{3}, \frac{(-1)^4}{4}, \dots\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\right\}$$

$$\{\sqrt{n}\}_{n=5}^{\infty} = \{\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \dots\}$$

What happens when n is very large?

$$\underbrace{(-1)^n = \begin{cases} 1 : n \text{ even} \\ -1 : n \text{ odd} \end{cases}}$$

$\lim_{n\to\infty} (-1)^n$ does not exist and it is not $\pm\infty$ either

divergent

$$\underbrace{\frac{(-1)^n}{n} = \begin{cases} 1/n : n \text{ even} \\ -1/n : n \text{ odd} \end{cases}}$$ both close to 0 for large n

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

convergent

large and positive

$\underbrace{\sqrt{n} \text{ is large when } n \text{ is large}}$

$$\lim_{n\to\infty} \sqrt{n} = \infty$$

The limit does not exist (as a number).

" $=\infty$ " means "large and positive"

divergent

Ex.

$$\lim_{n\to\infty} \frac{n^2+1}{3n^2+5} = \frac{1}{3}$$

Why? $$\lim_{n\to\infty} \frac{(n^2+1)/n^2}{(3n^2+5)/n^2} = \lim_{n\to\infty} \frac{1+1/n^2}{3+5/n^2} = \frac{1+0}{3+0} = \frac{1}{3}$$

Ex.

$$\lim_{n\to\infty} (2\ln(n) - \ln(n^2+1))$$

$$= \lim_{n\to\infty} (\ln(n^2) - \ln(n^2+1))$$

$$= \lim_{n\to\infty} \ln\left(\frac{n^2}{n^2+1}\right) = \ln\frac{1}{1} = 0$$

Why? $\lim_{n\to\infty} \ln\frac{n^2}{n^2+1} = \ln\frac{n^2/n^2}{(n^2+1)/n^2} = \ln\frac{1}{1+1/n^2} = \ln\frac{1}{1+0}$

Ex. $\lim_{n\to\infty} \frac{\ln n}{n} = ?$

$$\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty} = \left\{ \underset{0}{\overset{\|}{\frac{0}{1}}},\ \underset{.35}{\overset{\|}{\frac{\ln 2}{2}}},\ \underset{.37}{\overset{\|}{\frac{\ln 3}{3}}},\ \underset{.35}{\overset{\|}{\frac{\ln 4}{4}}},\right.$$

$$\underset{.32}{\overset{\|}{\frac{\ln 5}{5}}},\ \underset{.30}{\overset{\|}{\frac{\ln 6}{6}}},\ \underset{.28}{\overset{\|}{\frac{\ln 7}{7}}},\ \underset{.26}{\overset{\|}{\frac{\ln 8}{8}}} \ldots$$

Look at $\frac{\ln x}{x}$ For all reals $x > 0$.

If $\lim_{x\to\infty} \frac{\ln x}{x}$ exists or $= \infty$ or $= -\infty$, then

$\lim_{n\to\infty} \frac{\ln n}{n}$ will be the same

$$\lim_{x\to\infty} \frac{\ln x \ \}\to\infty}{x \ \}\to\infty} = \lim_{x\to\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0$$

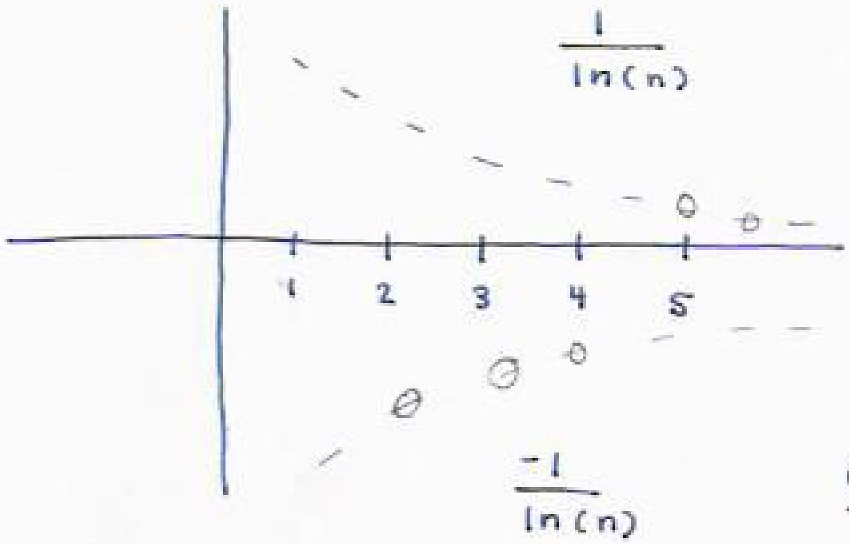
$\Rightarrow \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ $\qquad$ $\frac{0}{0}$ or $\pm\frac{\infty}{\infty} \Rightarrow$ use L'Hospital's Rule

Ex.

$$\lim_{n \to \infty} \frac{\cos(n)}{\ln(n)}$$

$$-1 \le \cos(n) \le 1 \Rightarrow \frac{-1}{\ln(n)} \le \frac{\cos n}{\ln(n)} \le \frac{1}{\ln(n)}$$

$$0 \leftarrow -\text{small} = \frac{-1}{\text{big}} \qquad \frac{1}{+\text{big}} = +\text{small} \to 0$$



$$\frac{\cos(n)}{\ln(n)} \qquad \lim_{n \to \infty} \frac{\cos(n)}{\ln(n)} = 0$$

Ex.

$$\lim_{n \to \infty} r^n = \begin{cases} 0 : & -1 < r < 1 \\ 1 : & r = 1 \\ \text{divergent and } = \infty : & r > 1 \\ \text{divergent} : & r \le -1 \end{cases}$$

r is constant

$$\lim_{n \to \infty} \left(\frac{-1}{3}\right)^n = 0 \qquad \text{Why?}$$

n	0	1	2	3	4
$\left(\frac{1}{-3}\right)^n$	1	$\frac{-1}{3}$	$\frac{1}{9}$	$\frac{-1}{27}$	$\frac{1}{81}$

For rigorous proof, use ln...

$$\lim_{n \to \infty} \ln \left| \left(\frac{-1}{3}\right)^n \right| = \lim_{n \to \infty} (-n \ln 3) = -\infty = -\text{big}$$

$$\Rightarrow \lim_{n \to \infty} \left| \left(\frac{-1}{3}\right)^n \right| = e^{-\text{big}} = \frac{1}{e^{+\text{big}}} = +\text{small} = 0$$

n is $+$ big $\Rightarrow (-3)^n$ is $\left\{ \begin{array}{ll} + \text{big} : & n \text{ even} \\ - \text{big} : & n \text{ odd} \end{array} \right\} \Rightarrow \lim\limits_{n \to \infty} (-3)^n$ is divergent

and $\neq \infty$

and $\neq -\infty$

You could say $\lim\limits_{n \to \infty} \left| (-3)^n \right| = \infty$

$$\lim_{n \to \infty} |-3|^n = \lim_{n \to \infty} 3^n = \infty$$

In general if $\lim\limits_{n \to \infty} |a_n| = 0$, then $\lim\limits_{n \to \infty} a_n = 0$

If $\lim\limits_{n \to \infty} |a_n| = \infty$, then $\lim\limits_{n \to \infty} a_n$ is divergent.

If $\lim\limits_{n \to \infty} |a_n|$ is divergent, then $\lim\limits_{n \to \infty} a_n$ is divergent.

If $\lim\limits_{n \to \infty} |a_n|$ is convergent, you need more info to say if
to $L \neq 0$
$\lim\limits_{n \to \infty} a_n$ converges.

Ex.

$$\{a_n\}_{n=2}^{\infty} = \left\{ \frac{-1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{-1}{8}, \frac{1}{9}, \frac{1}{10}, \dots \right\}$$

converges to 0 because

$$\{|a_n|\}_{n=2}^{\infty} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \dots \right\}$$

converges to 0.

Ex.

$\{b_n\} = \{5, -5, 5, -5, 5, -5 \dots\}$ diverges

$\{|b_n|\} = \{5, 5, 5, 5, 5, 5 \dots\}$ converges to 5.