Notes 4/19

Today $\begin{cases} 11.6\text{, part III: root test} \\ \text{HW due @ 5PM} \end{cases}$

Today / Tomorrow: Review of series tests

Consider $\underbrace{\sum_{n=0}^{\infty} r^n}_{\text{Geometric series}} = \begin{cases} \frac{1}{1-r} : & |r| < 1 \\ \text{divergent} : & |r| \geq 1 \end{cases}$

$$r = \frac{r^1}{r^0} = \frac{r^2}{r^1} = \frac{r^3}{r^2} = \frac{r^4}{r^3} = \dots$$

$$|r| = |r^1| = \sqrt{r^2} = \sqrt[3]{|r|^3} = \sqrt[4]{|r|^4} = \sqrt[5]{|r|^5} = \dots = \sqrt[n]{|r|^n} = \dots$$

Ratio Test	Root Test
$R = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	$R = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$
$R \geq 0$ always	$R \geq 0$ always
$\left.\begin{array}{l} R<1 \\ \text{or } R=\infty \end{array}\right\} \Rightarrow \sum_{n=0}^{\infty} \lvert a_n \rvert < \infty \Rightarrow \sum_{n=0}^{\infty} a_n$ converges	$\left.\begin{array}{l} R<1 \\ \text{or } R=\infty \end{array}\right\} \Rightarrow \sum_{n=0}^{\infty} \lvert a_n \rvert < \infty \Rightarrow \sum_{n=0}^{\infty} a_n$ converges
$R > 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges	$R > 1 \Rightarrow \sum_{n=0}^{\infty} a_n$ diverges
$R = 1 \Rightarrow$ inconclusive	$R = 1 \Rightarrow$ inconclusive

②

$$\sum_{n=2}^{\infty} \frac{1}{\ln^n(n)} = \sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$$

Ratio Test : $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is complicated!!

Root Test : $a_n = \frac{1}{(\ln n)^n} \Rightarrow \sqrt[n]{a_n} = \sqrt[n]{\frac{1}{(\ln n)^n}} = \frac{1}{\ln n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\ln n} = \frac{1}{+\text{ big}} \to 0$$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln^n n}$ converges

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} = \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{(n^2)}$$

$$|a_n| = a_n = \left(1+\frac{1}{n}\right)^{n^2} \Rightarrow \sqrt[n]{|a_n|} = \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \left(\left(1+\frac{1}{n}\right)^{n^2}\right)^{1/n}$$

$$= \left(1+\frac{1}{n}\right)^{(n^2\cdot\frac{1}{n})} = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \boxed{e} > 1 \Rightarrow \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} \text{ diverges}$$

* Why? $\lim_{n\to\infty} e^{\ln((1+\frac{1}{n})^n)} = e^{\lim_{n\to\infty} \ln((1+\frac{1}{n})^n)} = e^{\lim_{n\to\infty} n \ln(1+\frac{1}{n})}$ $\frac{d}{dn}$

$$= e^{\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n}) \to \ln(1+0)=0}{1/n \to 0}} \overset{\text{L'H rule}}{=} e^{\lim_{n\to\infty} \frac{\{(1+\frac{1}{n})^{-1}(-\frac{1}{n^2})\}}{\{-1/n^2\}}} = e^{\lim_{n\to\infty} (1+\frac{1}{n})^{-1}}$$

$\frac{d}{dn}\left(\frac{1}{n}\right)$

$$= e^{(1+0)^{-1}} = e^1 = e$$

on the other hand, if $b_n = \left(1-\frac{1}{n}\right)^{n^2}$ then $R = \lim_{n\to\infty} \sqrt[n]{|b_n|} = e^{-1} < 1$, so $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$ converges

③

$$\sum_{n=7}^{\infty}\left(\frac{(\ln n)(\sin n)}{n}\right)^n$$

$$|a_n| = \left|\frac{(\ln n)(\sin n)}{n}\right|^n = \left(\frac{\ln n}{n}|\sin n|\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{\ln n}{n}|\sin n|$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{\ln n}{n}|\sin n| = ?$$

$$-1 \le \sin n \le 1 \Rightarrow 0 \le |\sin n| \le 1$$

$$\Rightarrow \left(\frac{\ln n}{n}\right) 0 \le \left(\frac{\ln n}{n}\right)|\sin n| \le \left(\frac{\ln n}{n}\right)1$$

$$0 \le \frac{\ln n}{n}|\sin n| \le \frac{\ln n}{n}$$

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{\ln n}{n}|\sin n| \le \lim_{n\to\infty}\frac{\overset{\nearrow\infty}{\ln n}}{\underset{\searrow\infty}{n}} = \lim_{n\to\infty}\frac{1/n}{n} = 0$$

$$0 \le \lim_{n\to\infty}\frac{\ln n}{n}|\sin| \le 0$$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n]{|a_n|} = 0 < 1 \Rightarrow \sum_{n=7}^{\infty}\left|\left(\frac{(\ln n)(\sin n)}{n}\right)^n\right| \text{ Converges}$$

$$\Rightarrow \sum_{n=7}^{\infty}\left(\frac{(\ln n)(\sin n)}{n}\right)^n \text{ converges}$$

Failure of the root test

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$$

$$a_n = \left(1+\frac{1}{n}\right)^n \Rightarrow \sqrt[n]{|a_n|} = 1+\frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1+0 = 1 \Rightarrow \text{Root test inconclusive}$$

$$\lim_{n\to\infty} a_n = e \neq 0 \Rightarrow \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n \text{ diverges}$$

$$\sum_{n=1}^{\infty} \left(\frac{-1}{n^2+1}\right)^n \qquad a_n = \left(\frac{-1}{n^2+1}\right)^n \Rightarrow \sqrt[n]{|a_n|} = \frac{1}{n^2+1}$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{n^2+1} = \frac{1}{+\text{big}+1} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{-1}{n^2+1}\right)^n \text{ converges}$$

Alternate Solution:

$$\sum_{n=1}^{\infty} \left(\frac{-1}{n^2+1}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)^n} = \sum_{n=1}^{\infty} (-1)^n b_n$$

The Alternating series test says that if:

$$b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 \ldots \geq 0$$

then
$$\begin{cases} \lim_{n\to\infty} b_n = 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^n b_n \text{ converges } \checkmark \\ \lim_{n\to\infty} b_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^n b_n \text{ diverges} \end{cases}$$

← takes a bit of work

$$\lim_{n\to\infty} \left(\frac{1}{n^2+1}\right)^n = ?$$ Root test was easier...