Today. 11.8 & 11.9 (continued)

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Pg. 1

Last time: $S = \sum_{n=0}^{\infty} C_n (x-a)^n$

S converges if $|x-a| < R$;
S diverges if $|x-a| > R$, where $\frac{1}{R} = \lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = \lim_{n\to\infty} \sqrt[n]{|C_n|}$

R = "radius of convergance"

Generalization

$T = \sum_{n=0}^{\infty} C_n (g(x))^n$

T converges if $|g(x)| < R$

T diverges if $|g(x)| > R$

$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = \lim_{n\to\infty} \sqrt[n]{|C_n|}$

$\boxed{\text{Example}}$

$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{5^n} = \frac{x^1}{5^0} + \frac{x^3}{5^1} + \frac{x^5}{5^2} + \frac{x^7}{5^3} + \frac{x^9}{5^5} + \dots$

$x \sum_{n=0}^{\infty} \frac{1}{5^n} (x^2)^n$ $\quad C_n = \frac{1}{5^n}$

$\lim_{n\to\infty} \sqrt[n]{|C_n|}$

$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{5^n}}$

$= \lim_{n\to\infty} \frac{1}{5} = \frac{1}{5}$

$= \frac{1}{R} = \frac{1}{5} \Rightarrow R = 5$

$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{5^n}$ converges if $|x^2| < R = 5$ (same as $-\sqrt{5} < x < \sqrt{5}$)

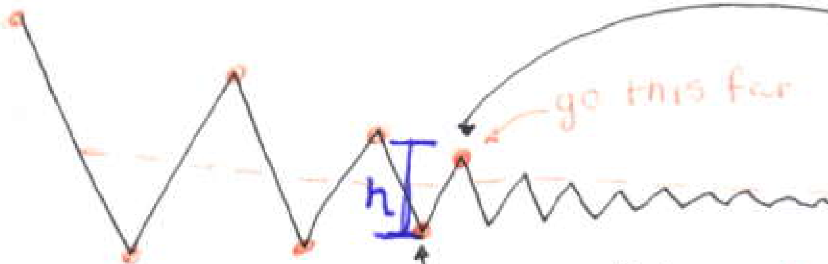
diverges if $|x^2| > R = 5$ (same as $x < -\sqrt{5}$ or $x > \sqrt{5}$)

$\boxed{\text{Last time:}}$

$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3^1} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \dots\right) = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$

Let's use this to estimate $\pi$ with an error at worst $\pm 0.00005$.

How many terms do we need? | Terms $n=0, \dots 8$ is 9 terms pg. 2




$$2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} \approx \boxed{3.1415997288\ldots}$$

Best Guess

adding terms for $\underbrace{n=0,\ldots,K}_{K+1 \text{ terms}}$

$$\boxed{3.1415687159} = 2\sqrt{3}\sum_{n=0}^{K-1}\frac{(-1)^n}{(2n+1)3^n}$$

adding terms for $\underbrace{n=0,\ldots,K-1}_{K \text{ terms}}$

$$=\frac{1}{2}(3.1415687159\ldots + 3.1415997788\ldots)$$

$= 3.1415842448767\ldots$

$|\text{error}| \le 0.000016$

$\pi = 3.14159265358\,98\ldots$

$\pi = 3.1415$

$$h = 2\sqrt{3}\sum_{n=0}^{K}\frac{(-1)^n}{(2n+1)3^n} - 2\sqrt{3}\sum_{n=0}^{K-1}\frac{(-1)^n}{(2n+1)3^n}$$

$$h = 2\sqrt{3}\left(\sum_{n=0}^{K}\frac{(-1)^n}{(2n+1)3^n} - \sum_{n=0}^{K-1}\frac{(-1)^n}{(2n+1)3^n}\right)$$

$$h = 2\sqrt{3}\,\frac{(-1)^K}{(2K+1)3^K}$$

Question: / Problem:
(Let's use this to estimate $\pi$ with an error at worst $\pm 0.00005$
How many terms do we need?) Terms $n=0,\ldots,8$ is 9 terms!

We want $-0.00005 \le \frac{h}{2} \le 0.00005$

Same as $\frac{|h|}{2} \le 0.00005$

$$0.00005 \ge \frac{|h|}{2} = \sqrt{3}\,\frac{1}{(2K+1)3^K}$$

You can't solve
$$0.00005 = \sqrt{3}\,\frac{1}{(2K+1)3^K}$$
exactly.

But we can find the least whole # $K$ that works by trial & error.

| K | h/2 |
|---|---|
| 5 | 0.000648 NO |
| 8 | 0.000016 ✓ |
| 7 | 0.000053 NO |

we need to use # all the way up to $K=8$

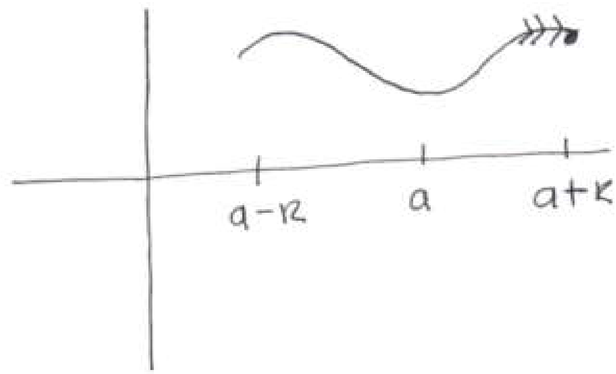
# Abel's Limit Theorem



If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$ has radius of convergence $R$,

and $f(a+R) = \sum_{n=0}^{\infty} C_n R^n$ converges, $R^n \longrightarrow (a+R-a)^n$

then $f(a+R) = \lim_{x \to (a+R)^-} f(x)$




From Thursday:

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges

$\underbrace{\text{for all } |x^2| < 1}_{\text{same as for all } |x|<1}$

because $\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|(-1)^n|} = 1 \Downarrow R = 1$

for any $t$ in $(-1,1)$

$$\tan^{-1} t = \int_0^t \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} \int_0^t (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}$$

$$\frac{\pi}{4} = \tan^{-1} 1 = \int_0^1 \frac{dx}{1+x^2} = \lim_{t\to 1^-} \int_0^t \frac{dx}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1}$$

Abel's Limit theorem applies if $\sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1}$ converges

converges to $\frac{\pi}{4}$

$\int_0^t x^{2n}\,dx = \frac{t^{2n+1}}{2n+1}$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots$