Today: ① area between curves
② start on arc-length

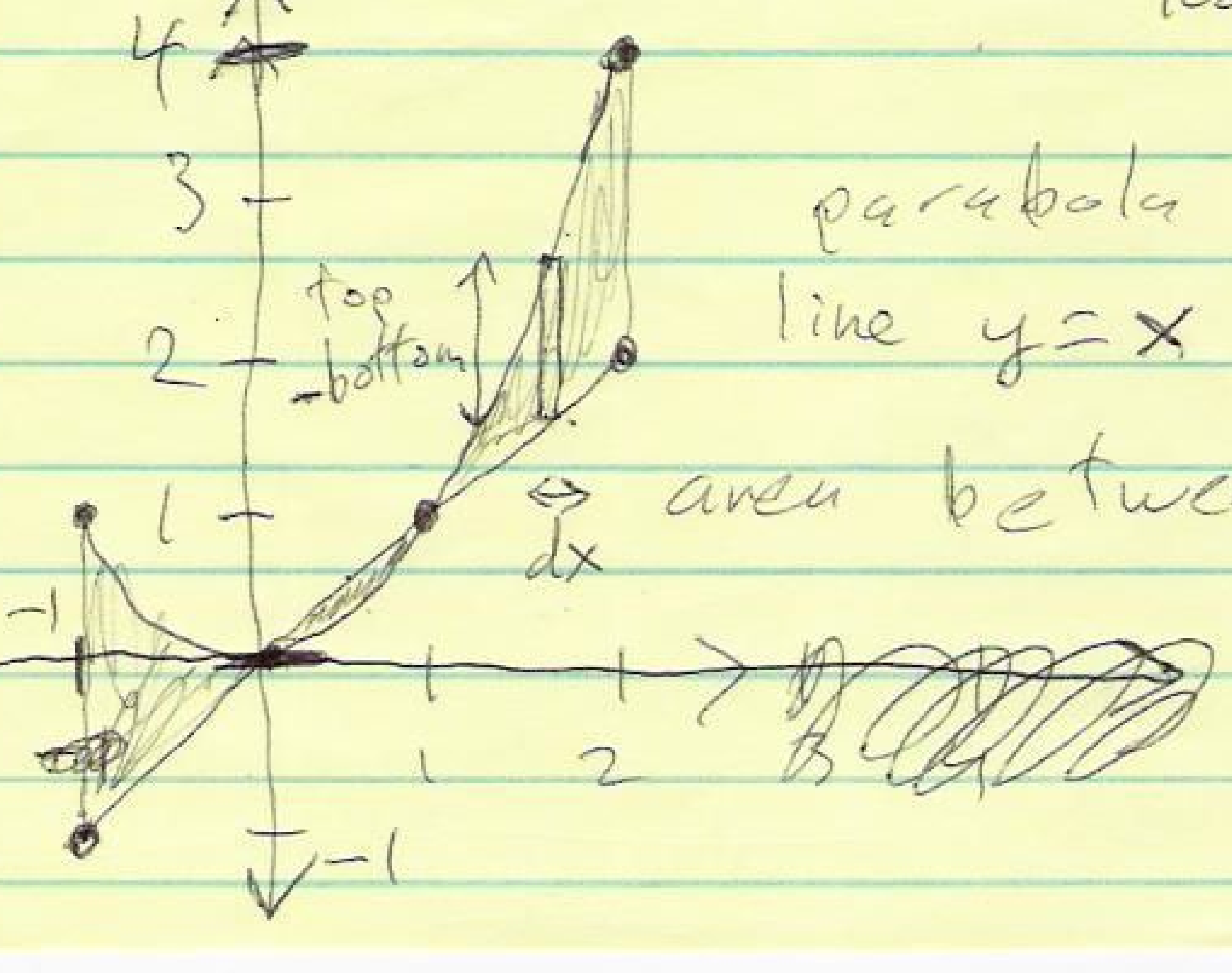
Not HW
① Find the area between the curves $y=x$ and $y=x^2$ over the x-values $-1 \le x \le 2$.

HW #1 Find area between the curves $y=x^2$ & $y=\sqrt{x}$ over $0 \le x \le 4$.

$y=x$ & $y=x^2$ $\quad -1 \le x \le 2$

Approach: $\int_{-1}^{2} (\text{top} - \text{bottom})\, dx$

Faster than $\int_{\text{loop}} y\, dx$, $\int_{\text{loop}} x\, dy$



parabola $y=x^2$
line $y=x$
area between curves

$$\text{area} = \int_{-1}^{0} (x^2 - x)\,dx + \int_0^1 (x - x^2)\,dx$$

$$+ \int_1^2 (x^2 - x)\,dx$$

$$= \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$+ \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_1^2$$

$$= (0 - 0) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)$$

$$+ \left(\frac{1^2}{2} - 1^3/3\right) - (0 - 0)$$

$$+ \left(2^3/3 - 2^2/2\right) - \left((1)^3/3 - 1^2/2\right)$$

$$= \cancel{\frac{1}{3}} + \overbrace{\frac{1}{2} + \frac{1}{2}}^{1} - \cancel{\frac{1}{3}} + \overbrace{\frac{8}{3} - \frac{6}{3}}^{2/3} - \frac{1}{3} + \frac{1}{2}$$

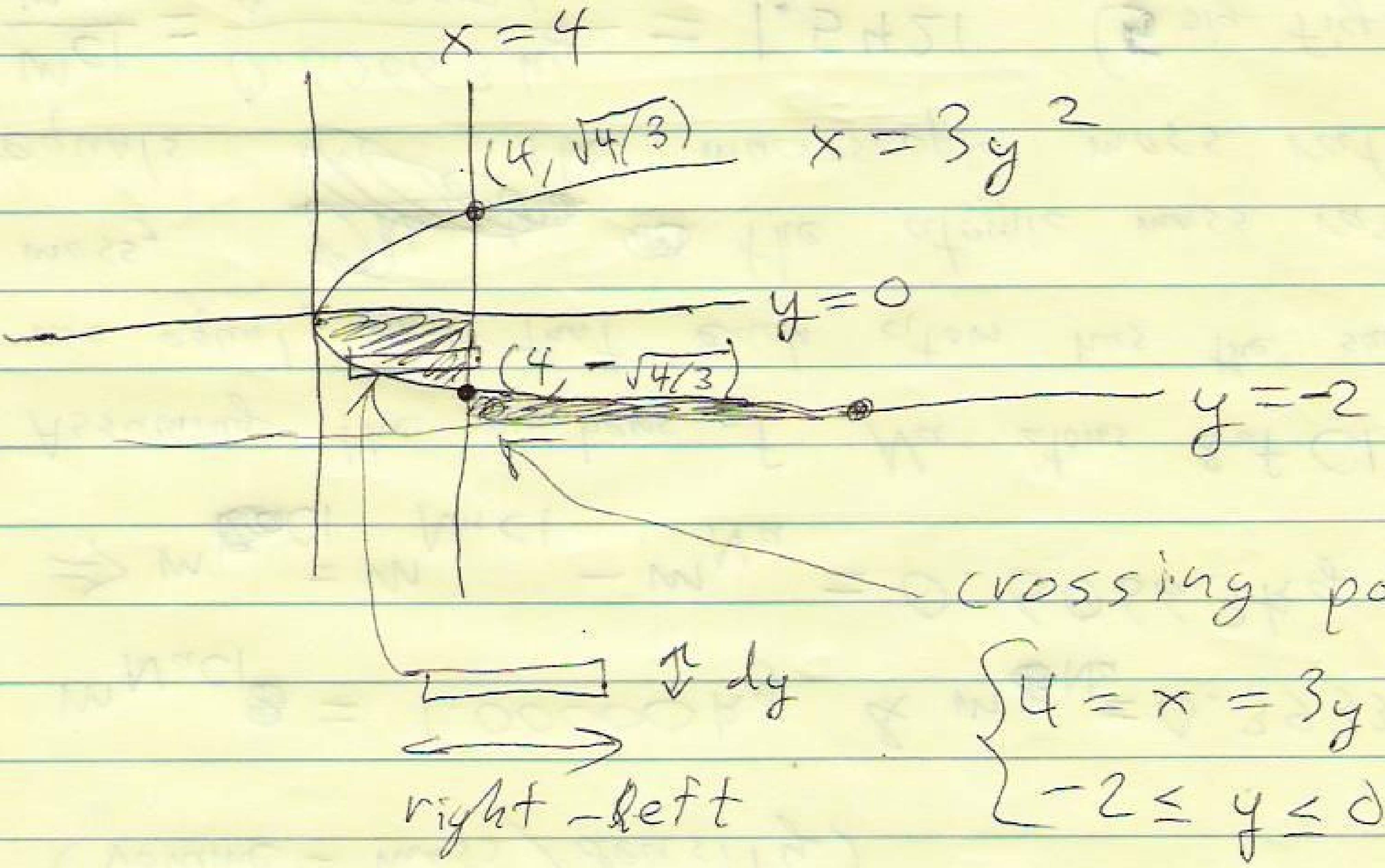
$$= 1 + \frac{2}{3} - \frac{1}{3} + \frac{1}{2} = \frac{11}{6}$$

"Sideways" example:

Find area between $x = 3y^2$

and ~~[illegible]~~ $x = 4$ over $-2 \le y \le 0$

Approach: $\int_{-2}^{0} (\text{right} - \text{left})\, dy$



crossing point:

$$\begin{cases} 4 = x = 3y^2 \\ -2 \le y \le 0 \end{cases}$$

$4 = 3y^2$

$4/3 = y^2$

$\pm\sqrt{4/3} = y$

$-\sqrt{4/3} = y$

area:

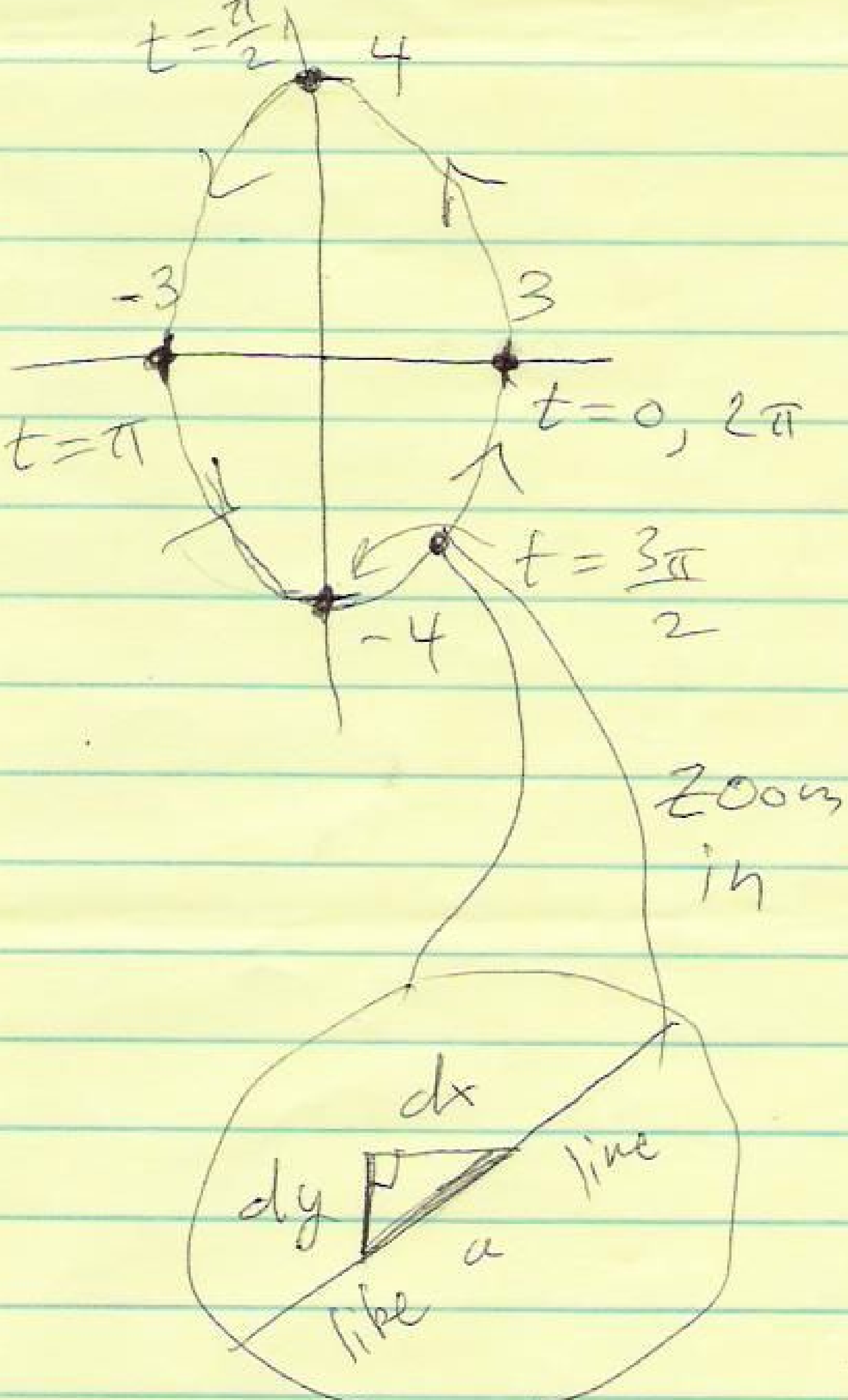
$$\int_{-2}^{-\sqrt{4/3}} (3y^2 - 4)\, dy$$

$$+ \int_{-\sqrt{4/3}}^{0} (4 - 3y^2)\, dy$$

HW#2 Use Simpson's Rule ($N=12$) to estimate the length of the elliptical spiral
$$\begin{cases} x = 2t\cos t \\ y = t\sin t \\ 0 \le t \le 4\pi \end{cases}$$

Circumference of ellipses:

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$



Parametrize:

$x = 3\cos t$

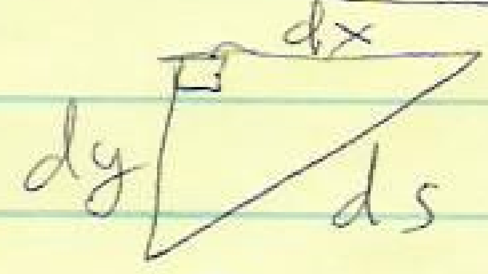
$y = 4\sin t$

$0 \le t \le 2\pi$

$dx^2 + dy^2 = ds^2$

$\sqrt{dx^2 + dy^2} = ds$

length of ellipse $= \displaystyle\int_{t=0}^{t=2\pi} ds$



s for arc length

$$x = 3\cos t \Rightarrow dx = (3\cos t)'\,dt$$
$$y = 4\sin t \Rightarrow dy = (4\sin t)'\,dt$$

$$dx = -3\sin t\,dt \Rightarrow dx^2 = 9\sin^2 t\,dt^2$$
$$dy = 4\cos t\,dt \Rightarrow dy^2 = 16\cos^2 t\,dt^2$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(9\sin^2 t + 16\cos^2 t)\,dt^2}$$

$$ds = \sqrt{9\sin^2 t + 16\cos^2 t}\;dt$$

$$\text{length} = \int_0^{2\pi} \sqrt{9\sin^2 t + 16\cos^2 t}\;dt$$

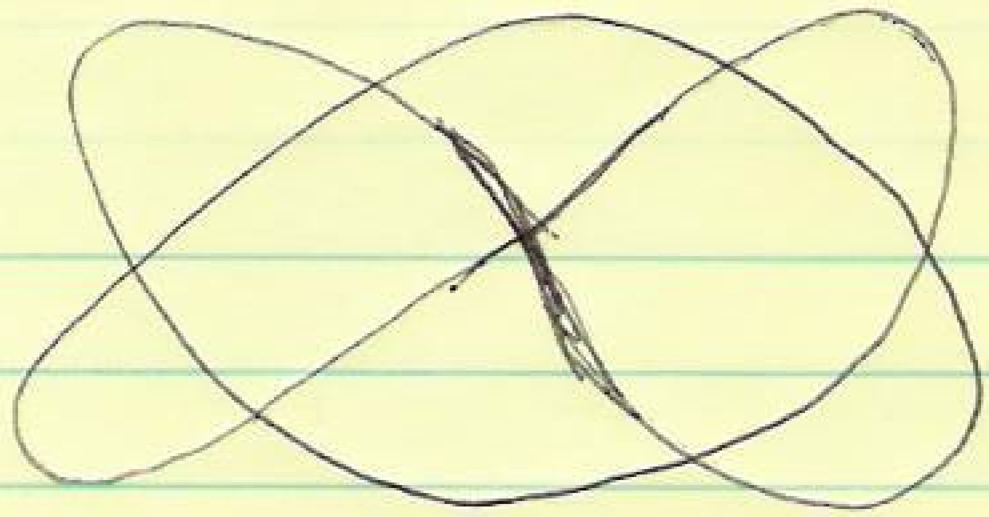
$$s(t) = \int_0^t ds$$

0 is arbitrary

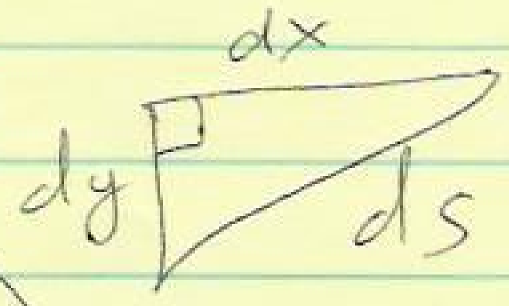
$$\text{length} = \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos^2 t) + 7\cos^2 t}\;dt$$

$$\text{length} = \int_0^{2\pi} \sqrt{9 + 7\cos^2 t}\;dt$$

~~$x = \cos 2t$~~
~~$y = \sin t$~~



$$\left.\begin{aligned} x &= \sin 2t \\ y &= \cos 3t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$



$x = \sin 2t$
$y = \cos 3t$

$$\text{length} = \int_0^{2\pi} ds$$

$dx = 2(\cos 2t)\,dt$
$dy = 3(-\sin 3t)\,dt$

$$ds^2 = dx^2 + dy^2$$

$dx^2 = 4\cos^2 2t\, dt^2$
$dy^2 = 9\sin^2 3t\, dt^2$

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{4\cos^2 2t + 9\sin^2 3t}\; dt$$

$$\text{length} = \int_0^{2\pi} \sqrt{4\cos^2 2t + 9\sin^2 3t}\; dt$$

$$\approx 15.3$$