8.8 (Keisler) $\rightarrow$ partial fractions

14.1 (Keisler) $\rightarrow$ "separable" differential equations

Lots of practice problems in 8.8 & 14.1

$$\frac{x^2 - 3x + 7}{(x^2 + x + 2)(4 - x)}$$

$x^2 + x + 2$ can't be factored (in the reals)

$a^2 + b^2$ can't be factored

$a^2 - b^2 = (a+b)(a-b)$

Is $x^2 + x + 2$ like $\underline{a^2+b^2}$ or a^2-b^2?

Complete the square...

$$\boxed{x^2 + px = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2} \quad \text{all } x, p$$

$$x^2 + 1x = \left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$x^2 + x + 2 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{8}{4}$$

$$= \left(x + \frac{1}{2}\right)^2 + \left(\sqrt{7/4}\right)^2$$

Another example: $\underline{x^2+2x-3}$ can be factored

$$x^2+2x = \left(x+\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2$$

$$x^2+2x-3 = (x+1)^2 - 1 - 3$$

$$x^2+2x-3 = (x+1)^2 - 2^2$$

$$x^2+2x-3 = (x+1+2)(x+1-2)$$

$$\underline{x^2+2x-3 = (x+3)(x-1)}$$

$$\frac{x^2+3x+7}{(x^2+x+2)(4-x)} = \frac{A_1x+B_1}{x^2+x+2} + \frac{C_1}{4-x}$$

$\uparrow$ new case

$$\frac{x^2+3x+7}{(x^2+2x-3)(4-x)} = \frac{A_2}{x+3} + \frac{B_2}{x-1} + \frac{C_2}{4-x}$$

$\uparrow$ old case

Look at new case:

$$\frac{x^2+3x+7}{(x^2+x+2)(4-x)} = \frac{Ax+B}{x^2+x+2} + \frac{C}{4-x}$$

$$x^2+3x+7 = (Ax+B)\underbrace{(4-x)}_{=0 \text{ if } x=4} + C\underbrace{(x^2+x+2)}_{\text{never } 0}$$

First, plug in $x=4$:

$$4^2+3(4)+7=0+C(4^2+4+2)$$

$$35=C(22) \qquad C=35/22$$

Pick 2 other x-values ...

How about, $x=0$, $x=1$?

$$0^2+3(0)+7=(A(0)+B)(4-0)+\frac{35}{22}(0^2+0+2)$$

$$7=B(4)+\frac{35}{11}$$

$$\frac{42}{11}=\frac{77}{11}-\frac{35}{11}=4B \Rightarrow \frac{21}{22}=B$$

$$1^2+3(1)+7=(A(1)+B)(4-1)+\frac{35}{22}(1^2+1+2)$$

$$11=\left(A+\frac{21}{22}\right)(3)+\frac{70}{11}$$

$$\frac{121}{11}=\left(A+\frac{21}{22}\right)(3)+\frac{70}{11} \Rightarrow \frac{41}{11}=\left(A+\frac{21}{22}\right)(3)$$

$$\frac{41}{33}=\left(A+\frac{21}{22}\right) \Rightarrow \frac{82}{66}\cancel{B}=A+\frac{63}{66} \Rightarrow A=\frac{19}{66}$$

$$\frac{x^2+3x+7}{(x^2+x+2)(4-x)}=\frac{\frac{19}{66}x+\frac{21}{22}}{x^2+x+2}+\frac{35/22}{4-x}$$

$$I=\int \frac{x^2+3x+7}{(x^2+x+2)(4-x)}\,dx=?$$

$$I=\underbrace{\int \frac{\frac{19}{66}x+\frac{21}{22}}{x^2+x+2}\,dx}_{J}+\int \frac{35/22}{4-x}\,dx$$

$$u=4-x$$
$$du=-dx$$
$$-du=dx$$

$$\int \frac{35/22}{4-x}\,dx=\int \frac{35/22}{u}(-du)=-\frac{35}{22}\int\frac{du}{u}$$

$$-\frac{35}{22}\ln|4-x|+c=-\frac{35}{22}\ln|u|+c$$

use our completion of the square

$$J=\int \frac{\frac{19}{66}x+\frac{21}{22}}{\left(x+\frac{1}{2}\right)^2+\left(\sqrt{7/4}\right)^2}\,dx=\int \frac{\frac{19}{66}x+\frac{21}{22}}{x^2+x+2}\,dx$$

$$\tan^2\theta+1=\sec^2\theta$$

First, $u=x+\frac{1}{2} \rightarrow u-\frac{1}{2}=x$

$du=dx$

$$J = \int \frac{\frac{19}{66}\left(u - \frac{1}{2}\right) + \frac{21}{22}}{u^2 + \left(\sqrt{7/4}\right)^2}\,du = \int \frac{\frac{19}{66}u - \frac{19}{132} + \frac{126}{132}}{u^2 + \left(\sqrt{7/4}\right)^2}\,du$$

$$J = \frac{19}{66}\int \frac{u\,du}{u^2 + \left(\sqrt{7/4}\right)^2} + \frac{107}{132}\int \frac{du}{u^2 + \left(\sqrt{7/4}\right)^2}$$

$w = u^2 + 7/4$

$dw = 2u\,du$

$dw/2 = u\,du$

need tangent substitution

$$J = \frac{19}{66}\int \frac{dw/2}{w} + \frac{107}{132}\int \frac{du}{(7/4)\left(\underbrace{4u^2/7}_{} + 1\right)}$$

$$\frac{4u^2}{7} = \tan^2\theta \qquad \underbrace{\tan^2\theta}_{\sec^2\theta}$$

Pick $\dfrac{2u}{\sqrt{7}} = \tan\theta$ & $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

$$\frac{2\,du}{\sqrt{7}} = \sec^2\theta\,d\theta$$

$$du = \frac{\sqrt{7}}{2}\sec^2\theta\,d\theta$$

$$J = \frac{19}{66}\left(\frac{1}{2}\right)\ln|w| + \frac{107}{132}\int \frac{(\sqrt{7}/2)\sec^2\theta\,d\theta}{(7/4)\sec^2\theta}$$

$$J = \frac{19}{132} \ln|u^2 + 7/4| + \frac{107}{132}\left(\frac{2}{\sqrt{7}}\right) \underbrace{\int d\theta}_{\theta + c}$$

(recall $u = x + \frac{1}{2}$)

$$\frac{2u}{\sqrt{7}} = \tan\theta \ \& \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\arctan\left(\frac{2u}{\sqrt{7}}\right) = \theta$$

$$J = \frac{19}{132} \ln\left|\left(x+\tfrac{1}{2}\right)^2 + \frac{7}{4}\right| + \frac{107}{66\sqrt{7}} \arctan\left(\frac{2x+1}{\sqrt{7}}\right) + c$$

$$\int \frac{x^2 + 3x + 7}{(x^2 + x + 2)(4-x)}\, dx$$

$$= \frac{19}{132} \ln\left|\left(x+\tfrac{1}{2}\right)^2 + \frac{7}{4}\right|$$

$$+ \frac{107}{66\sqrt{7}} \arctan\left(\frac{2x+1}{\sqrt{7}}\right)$$

$$- \frac{35}{22} \ln|4-x| + c$$

HW #1 $\int \frac{(2x+1)\,dx}{(x^2+5)(2+3x)} = ?$

#2 $\int_0^1 \frac{(2-x^2)\,dx}{(x^2+7x+1)(x+1)} = ?$