Taylor remainders that tend to 0:
(for Taylor expansions at $x=0$): 3 examples:
$\underbrace{\cos x, \quad \sin x,}_{\text{for all } x}$ $(1+x)^p$ (for $|x|<1$).

$\cos x$: $\quad R_n(x) = \int_0^x \frac{(x-t)^n}{n!} \cos^{(n+1)}(t)\,dt$

$\sin x$: $\quad R_n(x) = \int_0^x \frac{(x-t)^n}{n!} \sin^{(n+1)}(t)\,dt$

In the above 2 cases, $\cos^{(n+1)}(t)$ and $\sin^{(n+1)}(t)$ are of the form $\pm\cos t$ or $\pm\sin t$, so $|\cos^{(n+1)}(t)| \le 1$ and $|\sin^{(n+1)}(t)| \le 1$ because $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ for all θ.
Therefore, $|R_n(x)| \le \left|\int_0^x |(x-t)^n|\,dt\right| / n!$

$$= \frac{\left|\int_x^0 u^n\,(-du)\right|}{n!} = \frac{\left|\int_0^x u^n\,du\right|}{n!} = \frac{\left|[u^{n+1}/(n+1)]\big|_0^x\right|}{n!}$$

$$= \frac{|x^{n+1} - 0^{n+1}|}{n!\,(n+1)} = \frac{|x|^{n+1}}{(n+1)!} \qquad \boxed{u = x-t}$$

So, $R_n(x)$ is no bigger than $|x|^{n+1}/(n+1)!$
Therefore, if we can show that
$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ for all x, then the Taylor remainders $R_n(x)$ converge to 0 for $\cos(x)$ & $\sin(x)$ as $n\to\infty$, for all x.

Choose r to be the least integer $> |x|$.
Then, for all $n \ge r$,

$$0 \le \frac{|x|^n}{n!} = \frac{|x|}{1}\cdot\frac{|x|}{2}\cdot\frac{|x|}{3}\cdots\frac{|x|}{r-1}\cdot\frac{|x|}{r}\cdot\overbrace{\frac{|x|}{r+1}\cdots\frac{|x|}{n}}^{n-r \text{ terms}},$$

$$0 \le \frac{|x|^n}{n!} \le \frac{|x|^{r-1}}{(r-1)!} \cdot \frac{|x|}{r} \cdot \underbrace{\frac{|x|}{r} \cdots \frac{|x|}{r}}_{n-r \text{ terms}},$$

$$0 \le \frac{|x|^n}{n!} \le \frac{|x|^{r-1}}{(r-1)!} \left(\frac{|x|}{r}\right)^{n-r+1},$$

$$0 \le \frac{|x|^n}{n!} \le \frac{|x|^{r-1}|x|^{-r+1}}{(r-1)!\,r^{-r+1}} \left(\frac{|x|}{r}\right)^{n}$$

Since $0 \le |x| < r$, $0 \le \frac{|x|}{r} < 1$, so when n is large $\left(\frac{|x|}{r}\right)^n$ is small while $\frac{|x|^{r-1}|x|^{-r+1}}{(r-1)!\,r^{-r+1}} \left(= \frac{1}{(r-1)!/r^{r-1}} \right)$ does not change when n changes, so as n gets large, $\frac{|x|^{r-1}|x|^{-r+1}}{(r-1)!\,r^{-r+1}} \left(\frac{|x|}{r}\right)^n$ gets small, so $|x^n|/n!$ must also get small, so $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ indeed.

Hence, the Taylor remainders for the sine & cosine Taylor expansions at $x=0$ do converge to 0 as $n \to \infty$.

$(1+x)^p$:

$$R_n(x) = \int_0^x \frac{(x-t)^n}{n!} \underbrace{((1+t)^p)^{(n+1)}}_{(n+1)\text{th derivative}} dt$$

$$((1+t)^p)' = p(1+t)^{p-1}$$
$$((1+t)^p)'' = p(p-1)(1+t)^{p-2}$$
$$((1+t)^p)''' = p(p-1)(p-2)(1+t)^{p-3}$$
$$((1+t)^p)^{(k)} = p(p-1)(p-2)\cdots(p-(k-1))(1+t)^{p-k}$$
$$((1+t)^p)^{(n+1)} = p(p-1)(p-2)\cdots(p-n)(1+t)^{p-(n+1)}$$

Standard abbreviation:

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-(k-1))}{k!}$$

$$((1+t)^p)^{(n+1)}/n! = \frac{p(p-1)\cdots(p-n)}{n!}(1+t)^{p-n-1}$$

$$= \frac{p(p-1)\cdots(p-(n-1))}{n!}\cdot(p-n)(1+t)^{p-1}(1+t)^{-n}$$

$$= \binom{p}{n}(p-n)\frac{(1+t)^{p-1}}{(1+t)^n}$$

$$R_n(x) = \int_0^x \frac{(x-t)^n}{(1+t)^n}(1+t)^{p-1}\,dt \binom{p}{n}(p-n)$$

Let $t = xu$, so $dt = x\,du$,

$u = t/x$, $t=0 \Rightarrow u=0$, and
$t=x \Rightarrow u = x/x = 1$, so

$$R_n(x) = \int_0^1 \frac{(x-xu)^n}{(1+xu)^n}(1+xu)^{p-1}\, x\, du \binom{p}{n}(p-n)$$

$$= \binom{p}{n}(p-n)\, x^{n+1}\int_0^1 \frac{(1-u)^n}{(1+xu)^n}(1+xu)^{p-1}\,du$$

Now, <u>assume $-1<x<1$.</u>

Since $0 \le u \le 1$ for all values of u over which we integrate, either $xu=0$ (when $u=0$), or $-u<xu<u$ (when $u>0$). So, in either case, $-u \le xu \le u$.
Hence, $1-u \le 1+xu \le 1+u$. Since $u \le 1$, $-u \ge -1$, so $1-u \ge 1-1=0$, so actually $0 \le 1-u \le 1+xu$, so $0 \le \frac{1-u}{1+xu} \le \frac{1+xu}{1+xu} = 1$,

noting that $1+xu > 0$ because
$u=0 \Rightarrow 1+xu = 1+0 = 1 > 0$;
$u>0 \Rightarrow -u<xu \Rightarrow 1-u<1+xu$, and
$1 \ge u \Rightarrow -1 \le -u \Rightarrow 0=1-1 \le 1-u$, so
$1 \ge u > 0 \Rightarrow 0 \le 1-u < 1+xu$; thus,
$1 \ge u \ge 0 \Rightarrow 0 < 1+xu$.

Therefore, $0=0^n \le \left(\frac{1-u}{1+xu}\right)^n \le 1^n = 1$,

so $|R_n(x)| \le \left|\binom{p}{n}(p-n)\right| \int_0^1 (1+xu)^{p-1}\,du\ |x|^{n+1}$.

$\int_0^1 (1+xu)^{p-1}\,du$ does not depend

on n, so if we can show that

$$\lim_{n\to\infty} \left|\binom{p}{n}(p-n)\right| |x|^{n+1} = 0, \text{ then}$$

$$\lim_{n\to\infty} R_n(x) = 0 \text{ too.}$$

$$\left|\binom{p}{n}(p-n)\right||x|^{n+1} = \left|\frac{\overbrace{p(p-1)(p-2)\cdots(p-n)}^{n \text{ terms}}\overbrace{x\cdot x\cdot x\cdots x}^{n \text{ terms}}}{\underbrace{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n}_{n \text{ terms}}}\right|$$

$$= |px|\left|\frac{x(p-1)}{1}\right|\left|\frac{x(p-2)}{2}\right|\cdots\left|\frac{x(p-n)}{n}\right|$$

$$= |px|\left|x(p-1)\right|\left|x\left(\tfrac{p}{2}-1\right)\right|\cdots\left|x\left(\tfrac{p}{n}-1\right)\right|$$

$$= |px|\,|x(1-p)|\,|x(1-p/2)|\cdots|x(1-p/n)|$$

When n is large, $|1-p/n|^{-1} \approx |1|^{-1} = 1 > |x|$, so there is a (constant) integer l such that $|1-p/n|^{-1} > |x|$ for all $n \geq l$.

Therefore, when $n \geq l$:

$$\left|\binom{p}{n}(p-n)\right||x|^{n+1} = |px|\,|x(1-p)|\cdots\overbrace{\left|x\left(1-\tfrac{p}{l}\right)\right|\cdots\left|x\left(1-\tfrac{p}{n}\right)\right|}^{n-l+1 \text{ terms}}$$

$$\leq |px|\,|x(1-p)|\cdots\underbrace{\left|x\left(1-\tfrac{p}{l}\right)\right|\cdots\left|x\left(1-\tfrac{p}{l}\right)\right|}_{n-l+1 \text{ terms}}$$

$$= |px|\,|x(1-p)|\,|x(1-p/2)|\cdots|x(1-p/l)|\,|x(1-p/l)|^{n-l}$$

$$= |px| \underbrace{\frac{|x(1-p)| \cdots |x(1-p/\ell)|}{|x(1-p/\ell)|^{\ell}}}_{\text{does not depend on } n} \underbrace{|x(1-p/\ell)|^{n}}_{\text{small when } n \text{ is large because:}} = \text{small}$$

$$|1-p/\ell|^{-1} > |x| \Rightarrow 1 > |x||1-p/\ell| = |x(1-p/\ell)|.$$

So, $\lim_{n\to\infty} \left|\binom{p}{n}(p-n)\right| |x|^{n+1} = 0$,

so $\lim_{n\to\infty} R_n(x) = 0$ (when $-1<x<1$)