HW 45

①

$\langle P, Q \rangle$	$Q_x = \frac{\partial Q}{\partial x}$	$P_y = \frac{\partial P}{\partial y}$	gradient "everywhere"?
$\vec{A}$	$-y\sin(xy)$	$-x\sin(xy)$	no
$\vec{B}$	$\cos(xy) - xy\sin(xy)$	$\cos(xy) - xy\sin(xy)$	maybe
$\vec{C}$	$-y^2\sin(xy)$	$-x^2\sin(xy)$	no
$\vec{D}$	$4x^3-6x^2y$	$4x^3-6x^2y$	maybe
$\vec{E}$	$12xy^2-2y^3$	$4x^3-6x^2y$	no

($\vec{B}$, $\vec{D}$ have simply connected open domain $\mathbb{R}^2$; so, as we learned on Day 46, we can upgrade "maybe" to "yes.")

② $\vec{F} = \vec{\nabla}\left(\int x^{-1}dx + \int y^{-2}dy\right) = \vec{\nabla}\left(\ln x - \frac{1}{y}\right)$ in the first quadrant $\{(x,y) \mid x,y>0\}$, which contains I since $\sqrt{1+t^3} \geq 1$ & $\sqrt{t^5+1} \geq 1$ when $t \geq 0$. So, $\int_I \vec{F}\cdot d\vec{r}$

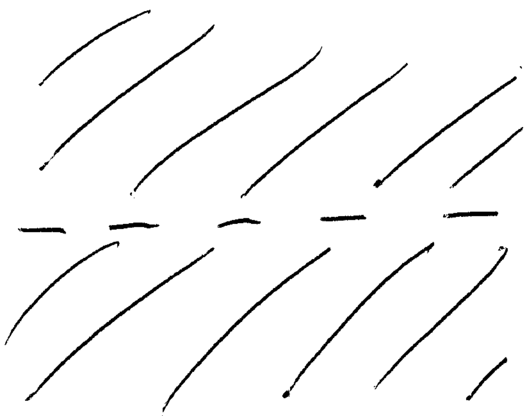
$$= \left(\ln x - \frac{1}{y}\right)\Big|_{t=0}^{t=1} = \left(\ln x - \frac{1}{y}\right)\Big|_{(x,y)=(1,1)}^{(x,y)=(\sqrt{2},\sqrt{2})} = \boxed{\ln\sqrt{2} + 1 - \frac{1}{\sqrt{2}}}$$

(3) $\int_J \vec{F}\cdot d\vec{r} = \left(\ln x - \frac{1}{y}\right)\Big|_{(1,2)}^{(1,2)} = 0$

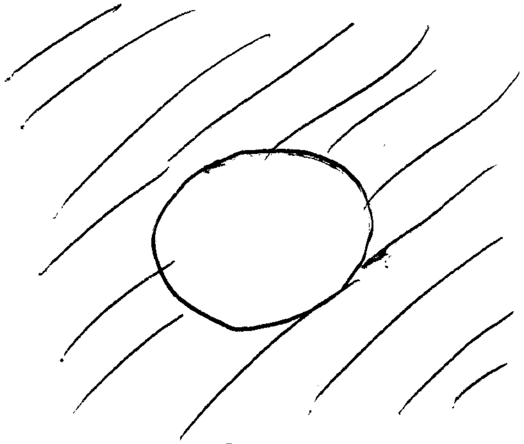
HW 45

Sketches to motivate HW46 answers:

HW 46

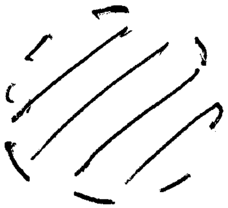


$y \neq 0$

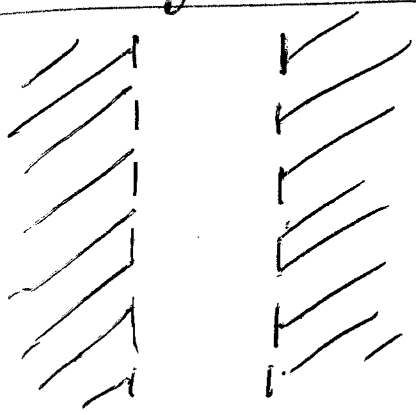


$y^2 \geq 5$

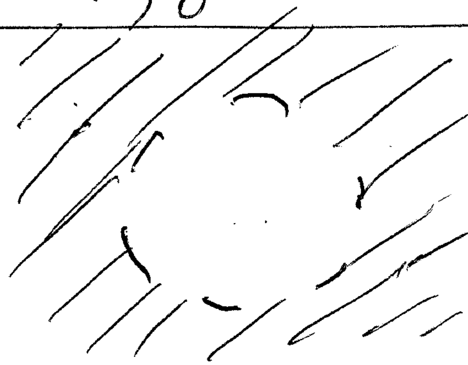
$x, y > 0$



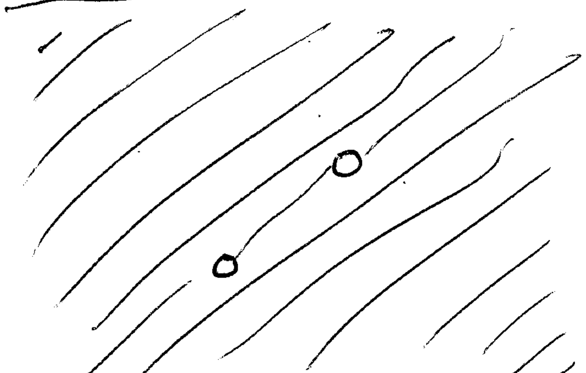
$r < 5$



$x^2 > 5$



$r > 5$



$(x,y) \neq (2,3), (3,4)$

Fill in the table with "yes" or "no". HW 46

Region	open?	connected?	simply connected?
$\{(x,y) \mid y \neq 0\}$	yes	no	no
$\{(x,y) \mid y^2 \geq 5\}$	no	yes	no
$\{(x,y) \mid x, y > 0\}$	yes	yes	yes
$\{(x,y) \mid r < 5\}$	yes	yes	yes
$\{(x,y) \mid x^2 > 5\}$	yes	no	no
$\{(x,y) \mid r > 5\}$	yes	yes	no
$\{(x,y) \mid (x,y) \neq (2,3), (3,4)\}$	yes	yes	no

It may help to sketch the region!

HW 47

① $f = \int P\,dx = 2x^3 + 5x^2y^4 - 3x + g(y)$

$\therefore Q = f_y = 0 + 20x^2y^3 + 0 + g'(y)$

$\therefore 1 = Q - 20x^2y^3 = g'(y)$ $\quad\therefore y + c = g(y)$

$\therefore \boxed{f(x,y) = 2x^3 + 5x^2y^4 - 3x + y + c}$

("$\therefore$" means "Therefore,")

② $7 = f(1,1) = 2 + 5 - 3 + 1 + c \Rightarrow c = 2$

$\therefore f(x,y) = 2x^3 + 5x^2y^4 - 3x + y + 2$

---

③ $f = \int P\,dx = \int \frac{-y^2\,dx}{x^2\sqrt{x^2+y^2}}$.

Let $x = |y|\tan\alpha$ (y constant).

$\therefore \begin{cases} dx = |y|\sec^2\alpha\,d\alpha \\ \sqrt{x^2+y^2} = \sqrt{(1+\tan^2\alpha)y^2} \end{cases}$

$= |y|\sec\alpha$

(for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$)

$f = -\int \frac{|y|y^2\sec^2\alpha\,d\alpha}{y^2\tan^2\alpha\,|y|\sec\alpha}$

$f = -\int \frac{\cos\alpha\,d\alpha}{\sin^2\alpha} \underset{\Uparrow}{=} -\int \frac{du}{u^2} = \frac{1}{u} + g(y)$

$u = \sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-(\sec^2\alpha)^{-1}} = \sqrt{1-(1+\tan^2\alpha)^{-1}} = \cdots$

HW 47

$$\cdots = \sqrt{(1+\tan^2\alpha - 1)(1+\tan^2\alpha)^{-1}} \underset{\substack{\Uparrow \\ x>0}}{=} \tan\alpha \Big/ \sqrt{1+\tan^2\alpha}$$

$$= |y| \tan\alpha \Big/ \sqrt{y^2+y^2\tan^2\alpha} = x \Big/ \sqrt{y^2+x^2} = x/r.$$

$$\therefore f(x,y) = \frac{1}{u} + g(y) = \frac{r}{x} + g(y).$$

$$\therefore \frac{y}{xr} = Q = f_y = g'(y) + \frac{\partial}{\partial y}\left(\frac{r}{x}\right).$$

$$\frac{\partial}{\partial y}\left(\frac{r}{x}\right) = \frac{1}{x}\frac{\partial}{\partial y}(x^2+y^2)^{\frac{1}{2}} = \frac{1}{2x}(x^2+y^2)^{\frac{1}{2}-1}(2y) = \frac{y}{x\sqrt{x^2+y^2}} = \frac{y}{xr}$$

$$\therefore g'(y) = 0. \qquad \text{So, } f(xy) = \frac{r}{x} + c = \frac{\sqrt{x^2+y^2}}{x} + c$$

$$\therefore 1 = f(1,0) = \frac{\sqrt{1^2+0^2}}{1} + c \qquad \therefore c = 0.$$

$$\boxed{\therefore f(x,y) = \frac{\sqrt{x^2+y^2}}{x} = \frac{r}{x} = \sec\theta}$$

(4) Use the substition $x = -|y| \tan\alpha$ this time.

HW 47

Still with $u = \sin\alpha$, $f(x,y) = \frac{-1}{u} + g(y) = \frac{r}{x} + g(y) = \frac{\sqrt{x^2+y^2}}{x} + c$

like in (3). $5 = f(-1,0) = \left(\sqrt{(-1)^2+0^2}/(-1)\right) + c \Rightarrow c = 6.$

So, $\boxed{f(x,y) = r/x + 6 = \sqrt{x^2+y^2}/x + 6 = \sec\theta + 6.}$

(5) $f = \int P\,dx = \int \frac{x\,dx}{(x^2+y^2)^{3/2}} \overset{\Uparrow}{=} \int \frac{du/2}{u^{3/2}} = \frac{1}{2}\int u^{-\frac{3}{2}}\,du = \underbrace{-u^{-\frac{1}{2}}}_{-(r^2)^{-1/2}} + g(y)$

$u = x^2+y^2$ (y constant)
$du = 2x\,dx$

$= -\frac{1}{r} + g(y)$. $\therefore y/r^3 = Q = f_y = g'(y) - \frac{\partial}{\partial y}(x^2+y^2)^{-\frac{1}{2}}$

$= g'(y) - \left(-\frac{1}{2}\right)(x^2+y^2)^{-\frac{1}{2}-1}(2y) = g'(y) + y/r^3$. $\therefore g'(y) = 0$.

$\therefore f = -\frac{1}{r} + c$. These solutions are well-defined & differentiable everywhere except (0,0). So, $\boxed{\text{yes}}$;

$F(x,y) = \frac{-1}{\sqrt{x^2+y^2}}$ (choosing $c=0$, for example) is a solution to $\vec{\nabla} f = \langle x,y\rangle / r^3$ on all of $\mathbb{R}^2 - \{(0,0)\}$.

① $\iint_D (Q_x - P_y)\,dA = \int_{\partial D}(P\,dx + Q\,dy)$ (Green's Thm.) $\boxed{\text{HW } 48}$

P	Q	$Q_x - P_y$
0	x	$1-0$
0	$x^2/2$	$x-0$
0	xy	$y-0$

$$x_{cm} = \frac{\iint_D x\,dA}{\iint_D 1\,dA} = \frac{\int_{\partial D} x^2\,dy/2}{\int_{\partial D} x\,dy}$$

$$y_{cm} = \frac{\iint_D y\,dA}{\iint_D 1\,dA} = \frac{\int_{\partial D} xy\,dy}{\int_{\partial D} x\,dy}$$

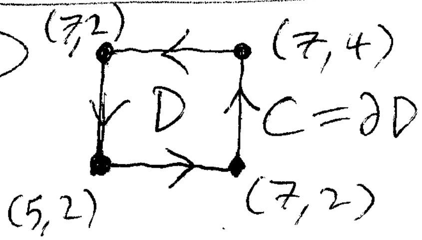
$$\int_{\partial D} x\,dy = \int_0^1 (t-t^3)(2t-1)\,dt = 1/60$$

$$\frac{1}{2}\int_{\partial D} x^2\,dy = \frac{1}{2}\int_0^1 (t-t^3)^2(2t-1)\,dt = 1/280$$

$$\int_{\partial D} xy\,dy = \int_0^1 (t-t^3)(t^2-t)(2t-1)\,dt = -1/420$$

$$\Rightarrow \begin{cases} x_{cm} = 3/14 \\ y_{cm} = -1/7 \end{cases}$$

②



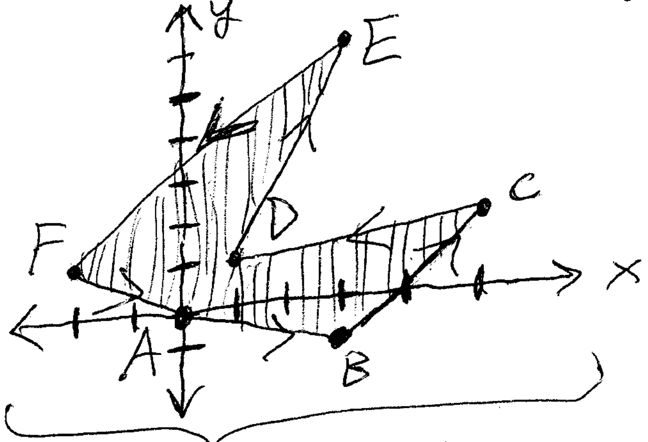
$$\iint_D = \int_{x=5}^{x=7}\int_{y=2}^{y=4} \qquad \int_C \left(\frac{dx}{y} - \frac{dy}{x}\right)$$

$$= \int_{\partial D}(y^{-1}dx - x^{-1}dy) = \iint_D \left((-x^{-1})_x - (y^{-1})_y\right) dA$$

$$= \int_5^7 \left(\int_2^4 (x^{-2}+y^{-2})\,dy\right) dx = \boxed{\frac{43}{70}}$$

HW 49

① $\text{area}(ABCDEF) = \int_{\underbrace{ABCDEFA}_{\text{counterclockwise boundary loop}}} x\,dy$ by Green's Thm.



Hexagon H

$\partial H = ABCDEFA$

$= AB + BC + CD + DE + EF + FA$

segment	AB	BC	CD	DE	EF	FA
x	$3t$	$3+2t$	$5-4t$	$1+2t$	$3-5t$	$-2+2t$
dy	$-dt$	$3dt$	$-dt$	$5dt$	$-5dt$	$-dt$

parametrizations each from $t=0$ to $t=1$.

$$\text{area} = \int_0^1 (-3t + 9 + 6t - 5 + 4t + 5 + 10t - 15 + 25t + 2 - 2t)\,dt$$

$$= \int_0^1 (-4 + 40t)\,dt = -4 + 20 = \boxed{16}$$

② $$x_{cm} = \frac{1}{16}\iint_H x\,dA = \frac{1}{16}\int_{\partial H} \frac{x^2}{2}\,dy = \frac{1}{32}\int_0^1 \Big[(3t)^2(-1) + (3+2t)^2(3)$$
$$+ (5-4t)^2(-1) + (1+2t)^2(5) + (3-5t)^2(-5) + (-2+2t)^2(-1)\Big]dt = \boxed{\frac{133}{96}}$$

segment	AB	BC	CD	DE	EF	FA
y	$-t$	$3t-1$	$2-t$	$1+5t$	$6-5t$	$1-t$

$\Rightarrow y_{cm} = \frac{1}{16}\iint_H y\,dA = \frac{1}{16}\int_{\partial H} xy\,dy$

$$= \frac{1}{16}\int_0^1 \Big[(3t)(-t)(-1) + (3+2t)(3t-1)(3) + (5-4t)(2-t)(-1) + \cdots\Big]dt = \boxed{\frac{73}{48}}$$

HW 49

$$\iint_E xy\,dx\,dy = \iint_E \left(\frac{\partial}{\partial x}(x^2y/2) - \frac{\partial}{\partial y}(0)\right) dA$$

$$= \int_{\partial E} \left(0\,dx + \tfrac{1}{2}x^2y\,dy\right)$$

$$= \left(\int_I x^2y\,dy + \int_J x^2y\,dy + \int_K x^2y\,dy\right)/2$$

$$= \frac{1}{2}\int_0^{2\pi} \Big(\left[(3+(6+\sin(7t))\cos(t)\right]^2 \left[(4+2\sin(7t))\sin(t)\right]\left[14\cos(7t)\sin(t) + (4+2\sin(7t))\cos(t)\right]$$

$$+ \left[1+2\cos(t)\right]^2\left[-\sin(t)\right]\left[-\cos(t)\right]$$

$$+ \left[5+\cos(t)\right]^2\left[1-2\sin(t)\right]\left[-2\cos(t)\right] \Big)\, dt$$

$$= \boxed{-20\pi}$$