(HW26)

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = QR$$

$$Q = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & \sqrt{2/3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

If building $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ manually, you'll get the same:

$$\vec{w}_1 = \vec{u}_1 \qquad \vec{w}_2 = \vec{u}_2 - \frac{\vec{w}_1 \cdot \vec{u}_2}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 \qquad \vec{w}_3 = \vec{u}_3 - \frac{\vec{w}_1 \cdot \vec{u}_3}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{w}_2 \cdot \vec{u}_3}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2$$

$$\vec{v}_1 = \frac{\vec{w}_1}{\sqrt{\vec{w}_1 \cdot \vec{w}_1}} \qquad \vec{v}_2 = \frac{\vec{w}_2}{\sqrt{\vec{w}_2 \cdot \vec{w}_2}} \qquad \vec{v}_3 = \frac{\vec{w}_3}{\sqrt{\vec{w}_3 \cdot \vec{w}_3}}.$$

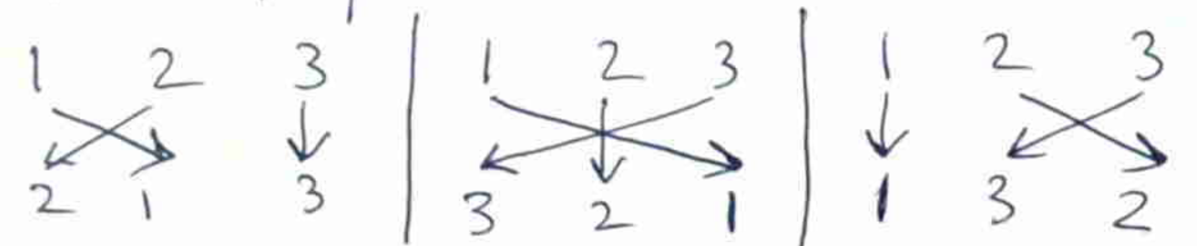
P.S. The $R = \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$ can be interpreted

as follows:
$$\begin{cases} \vec{u}_1 = \sqrt{3}\,\vec{v}_1 + 0\,\vec{v}_2 + 0\,\vec{v}_3 \\ \vec{u}_2 = \frac{1}{\sqrt{3}}\vec{v}_1 + \sqrt{\frac{2}{3}}\,\vec{v}_2 + 0\,\vec{v}_3 \\ \vec{u}_3 = \frac{1}{\sqrt{3}}\vec{v}_1 + -\frac{1}{\sqrt{6}}\vec{v}_2 + \frac{1}{\sqrt{2}}\vec{v}_3 \end{cases}$$

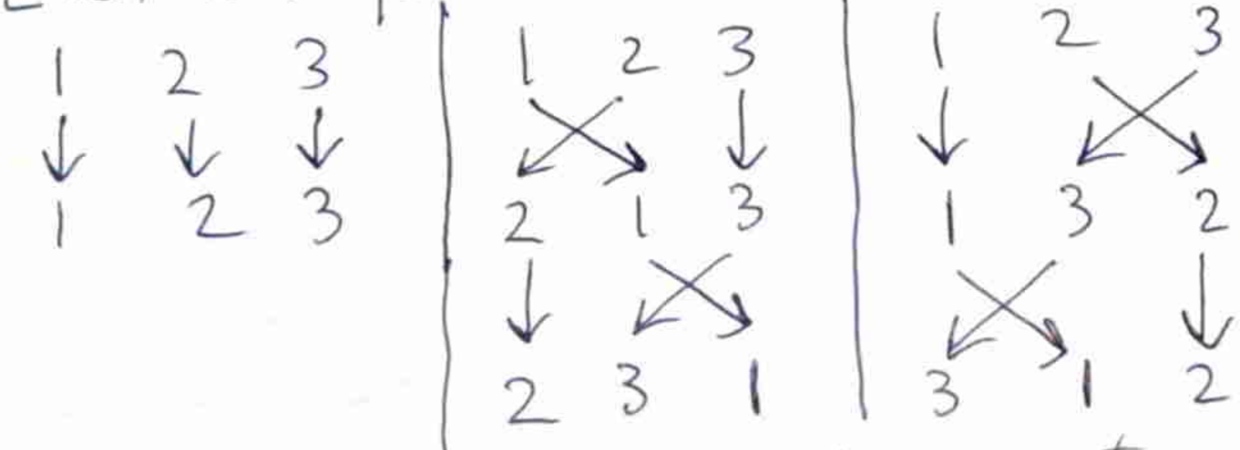
But we don't need this for HW26.

$$\det(Q) = Q_{11}Q_{22}Q_{33} + Q_{12}Q_{23}Q_{31} + Q_{13}Q_{21}Q_{32}$$
$$- Q_{12}Q_{21}Q_{33} - Q_{13}Q_{22}Q_{31} - Q_{11}Q_{23}Q_{32}$$

Why? Odd (−) permutations (made with 1 swap):



Even (+) permutations (made with 0 or 2 swaps):



(213, 321, 132, 123, 231, 321 are only ways to permute 123.)

($Q_{i,j}$ = entry of Q in row i & column j.)

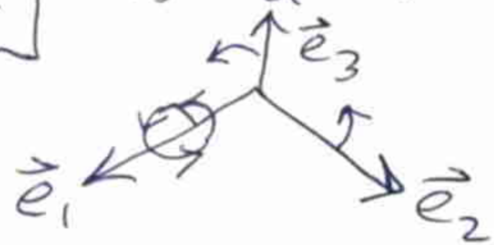
$\det(Q) = 1$. (Use a calculator to get the determinant the fast way!)

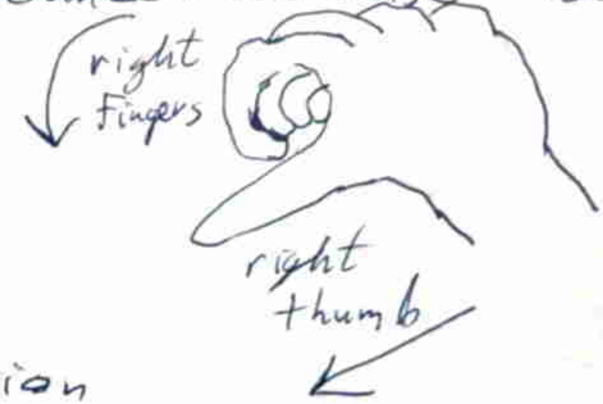
Since $\det(Q) > 0$, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is "right-handed."

(You can check $\vec{v}_1 \times \vec{v}_2 = \vec{v}_3$ if you know about cross products.)

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\pi/6 & -\sin\pi/6 \\ 0 & \sin\pi/6 & \cos\pi/6 \end{bmatrix}$$ is a 30° "counterclockwise" rotation

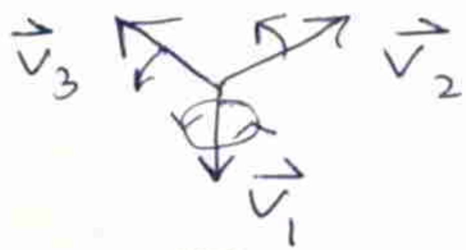
around $\vec{e}_1$:





QHQ^{-1} is a 30° ccw rotation

around $\vec{v}_1$:



$$QHQ^{-1} = QHQ^T = \begin{bmatrix} (1+\sqrt{3})/3 & (1-\sqrt{3})/3 & 1/3 \\ 1/3 & (1+\sqrt{3})/3 & (1-\sqrt{3})/3 \\ (1-\sqrt{3})/3 & 1/3 & (1+\sqrt{3})/3 \end{bmatrix}$$

$\Uparrow$

Q is orthogonal

(A decimal approximation is an acceptable answer too.)

① $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -2 \\ -2 & -5 \end{bmatrix}$ (HW27

$\det(A - I_3\lambda) = (1-\lambda)(4-\lambda)(6-\lambda) + 2\cdot5\cdot3 + 3\cdot2\cdot5$
$- (1-\lambda)\cdot5\cdot5 - 3\cdot(4-\lambda)\cdot3 - 2\cdot2\cdot(6-\lambda) = -\lambda^3 + 11\lambda^2 + 4\lambda - 1$

The exact formulas for the roots of this polynomial are quite complicated. I recommend using a calculator to get numerical approximations of the roots. (Even a calculator without a "solve" command can be used to find approx. roots using Newton's method for solving $f(x) \approx 0$: guess x_0 and use $x_{n+1} = x_n - f(x_n)/f'(x_n)$ to find better estimates.)

$\lambda_1 = 0.170915\ldots$ $\lambda_2 = -0.515729\ldots$ $\lambda_3 = 11.344814\ldots$

To find eigenvectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ corresponding to $\lambda_1, \lambda_2, \lambda_3$, we need bases $\{\vec{u}_1\}, \{\vec{u}_2\}, \{\vec{u}_2\}$ for $\ker(A - \lambda_1 I_3)$, $\ker(A - \lambda_2 I_3)$, and $\ker(A-\lambda_3 I_3)$. To find these numerically, delete the bottom row of each $A-\lambda_j I_3$ and then use Gauss-Jordan elimination:

$A-\lambda_1 I_3$

$$\begin{bmatrix} 0.8291 & 2 & 3 \\ 2 & 3.8291 & 5 \\ \cancel{3} & \cancel{5} & \cancel{5.8291} \end{bmatrix}$$

RREF:

$$\begin{bmatrix} 1 & 0 & -1.8019 \\ 0 & 1 & 2.2470 \end{bmatrix}$$

$A-\lambda_2 I_3$

$$\begin{bmatrix} 1.5157 & 2 & 3 \\ 2 & 4.5157 & 5 \\ \cancel{3} & \cancel{5} & \cancel{6.5157} \end{bmatrix}$$

RREF:

$$\begin{bmatrix} 1 & 0 & 1.2470 \\ 0 & 1 & 0.5550 \end{bmatrix}$$

$A-\lambda_3 I_3$

$$\begin{bmatrix} -10.3448 & 2 & 3 \\ 2 & -7.3448 & 5 \\ \cancel{3} & \cancel{5} & \cancel{-5.3448} \end{bmatrix}$$

RREF:

$$\begin{bmatrix} 1 & 0 & -0.4450 \\ 0 & 1 & -0.8019 \end{bmatrix}$$

This works because $\det(A-\lambda_j I_3)=0$ implies that one of the rows of $A-\lambda_j I_3$ is a linear combination of the others and so represents a redundant equation. Usually, every row is a lin. comb. of the others, so you can remove the bottom row. If we don't remove a row, the fact that we're only using approx. eigenvalues will produce a useless RREF corresponding to a kernel of $\{\vec{0}\}$.

Interpreting each row $[a \ b \ c]$ as $ax_1+bx_2+cx_3=0$, we obtain the following three systems.

$x_1 = 1.18019x_3$	$x_1 = -1.2470x_3$	$x_1 = 0.4450x_3$
$x_2 = -2.2470x_3$	$x_2 = -0.5550x_3$	$x_2 = 0.8019x_3$
$x_3 = x_3$ (free)	$x_3 = x_3$ (free)	$x_3 = x_3$ (free)

Letting $x_3 = 1$ (so that $\vec{x} \neq \vec{0}$), we obtain our bases:

$$\{\vec{u}_1\} \approx \left\{ \begin{bmatrix} 1.8019 \\ -2.2470 \\ 1 \end{bmatrix} \right\} \quad \Bigg| \quad \{\vec{u}_2\} \approx \left\{ \begin{bmatrix} -1.2470 \\ -0.5550 \\ 1 \end{bmatrix} \right\} \quad \Bigg| \quad \{\vec{u}_3\} \approx \left\{ \begin{bmatrix} 0.4450 \\ 0.8019 \\ 1 \end{bmatrix} \right\}$$

Finally, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is orthogonal by the Spectral Theorem (and we can check $\vec{u}_1 \cdot \vec{u}_2 \approx \vec{u}_2 \cdot \vec{u}_3 \approx \vec{u}_3 \cdot \vec{u}_1 \approx 0$), so, to get an orthonormal basis of eigenvectors, we use

$$\vec{v}_1 = \frac{\vec{u}_1}{\sqrt{\vec{u}_1 \cdot \vec{u}_1}} \approx \begin{bmatrix} 0.5910 \\ -0.7370 \\ 0.3280 \end{bmatrix} \qquad \vec{v}_2 = \frac{\vec{u}_2}{\sqrt{\vec{u}_2 \cdot \vec{u}_2}} \approx \begin{bmatrix} -0.7370 \\ -0.3280 \\ 0.5910 \end{bmatrix}$$

$$\vec{v}_3 = \frac{\vec{u}_3}{\sqrt{\vec{u}_3 \cdot \vec{u}_3}} \approx \begin{bmatrix} 0.3280 \\ 0.5910 \\ 0.7370 \end{bmatrix}$$

(We can check our work by verifying that $\vec{v}_j \cdot \vec{v}_k$ is ≈ 1 when $j=k$ and ≈ 0 when $j \neq k$, and that $A\vec{v}_j \approx \lambda_j \vec{v}_j$.) Note: The order of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ doesn't matter, nor do -1 factors: $\pm v_i, \pm v_j, \pm v_k$ where $i \neq j \neq k \neq i$ is the general correct form.

For B, we proceed as with A, but now the roots are easier to find: $0 = \det\left(\begin{bmatrix} 3-\lambda & -2 \\ -2 & -5-\lambda \end{bmatrix}\right) = (3-\lambda)(-5-\lambda) - (-2)(-2)$

$= \lambda^2 + 2\lambda - 19 = (\lambda+1)^2 - 20 \iff \lambda = -1 \pm \sqrt{20} = -1 \pm 2\sqrt{5}$.

Let $\lambda_1 = -1 - 2\sqrt{5}$ & $\lambda_2 = -1 + 2\sqrt{5}$.

$\vec{x} \in \ker(B - \lambda_j I_2) \iff (3-\lambda_j)x_1 + (-2)x_2 = 0$ $\left(\begin{bmatrix} 3-\lambda_j & -2 \end{bmatrix}\right)$

$\iff$ (using $R_1 \to R_1/(3-\lambda_j)$) $x_1 - \frac{2}{3-\lambda_j}x_2 = 0$ $\left(\begin{bmatrix} 1 & \frac{-2}{3-\lambda_j} \end{bmatrix}\right)$.

Choosing $x_2 = 3 - \lambda_j$ (any $x_2 \neq 0$ will do), $\vec{x} = \begin{bmatrix} 2 \\ 3-\lambda_j \end{bmatrix}$ is a basis for $\ker(B - \lambda_j I_2)$. Dividing $\vec{x}$ by $\sqrt{\vec{x} \cdot \vec{x}}$, we obtain $\vec{v}_j = \frac{1}{\sqrt{4 + (3-\lambda_j)^2}} \begin{bmatrix} 2 \\ 3-\lambda_j \end{bmatrix} \approx \begin{bmatrix} 0.2298 \\ 0.9732 \end{bmatrix}, \begin{bmatrix} 0.9732 \\ -0.2298 \end{bmatrix}$

HW28

$$0=\det\Big(\underbrace{\begin{bmatrix}1-\lambda & 4\\ -3 & 1-\lambda\end{bmatrix}}_{A-\lambda I_2}\Big)=(1-\lambda)^2+12$$

$\iff \lambda = 1 \pm 2\sqrt{3}\,i.$ $\quad \ker(A-\lambda_j I_2)=\ker\big(\begin{bmatrix}\pm 2\sqrt{3}\,i & 4\end{bmatrix}\big)$

$$=\operatorname{span}\left\{\begin{bmatrix}4\\ \mp 2\sqrt{3}\,i\end{bmatrix}\right\}$$

Let $\begin{cases}\lambda_1 = 1-2\sqrt{3}\,i\\ \lambda_2 = 1+2\sqrt{3}\,i\end{cases}$.

Let $\vec{v}_1=\begin{bmatrix}4\\ -2\sqrt{3}\,i\end{bmatrix}$ & $\vec{v}_2=\begin{bmatrix}4\\ +2\sqrt{3}\,i\end{bmatrix}$.

(Check that $A\vec{v}_1=\lambda_1\vec{v}_1$ & $A\vec{v}_2=\lambda_2\vec{v}_2$.)

$$\frac{d}{dt}\begin{bmatrix}x\\ y\end{bmatrix}=A\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}^{-1}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\iff \frac{d}{dt}\begin{bmatrix}r\\ s\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}r\\ s\end{bmatrix}=\begin{bmatrix}\lambda_1 r\\ \lambda_2 s\end{bmatrix}$$ where

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}. \quad \text{So, } \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = r\vec{v}_1 + s\vec{v}_2$

$$= \begin{bmatrix} 4c_1 e^{(1-2\sqrt{3}i)t} + 4c_2 e^{(1+2\sqrt{3}i)t} \\ -2\sqrt{3}ic_1 e^{(1-2\sqrt{3})t} + 2\sqrt{3}ic_2 e^{(1+2\sqrt{3}i)t} \end{bmatrix}$$ for $c_1, c_2 \in \mathbb{C}$.

P.S. The general real solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^t \begin{bmatrix} 4\,(b_1 \cos(2\sqrt{3}t) + b_2 \sin(2\sqrt{3}t)) \\ 2\sqrt{3}\,(b_2 \cos(2\sqrt{3}t) - b_1 \sin(2\sqrt{3}t)) \end{bmatrix}$$ for $b_1, b_2 \in \mathbb{R}$

and is found by setting $2c_1 = b_1 + b_2 i$ & $2c_2 = b_1 - b_2 i$.

(2) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ since $\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 \geq 1 > 0$ for $\lambda \in \mathbb{R}$.

Any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad \neq bc$ will be invertible and any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $(a-d)^2 + 4bc < 0$ will have no real eigenvalues.

(3) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ since $\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2$.

Any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $(a-d)^2 + 4bc = 0$ will have exactly one real eigenvalue (and be invertible if also $ad \neq bc$).

HW29

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 3 & 0 \\ 1 & 5 & 3 \end{bmatrix} = HJH^{-1} \quad \text{where} \quad H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 20 & 1 & 0 \end{bmatrix}$$

and $J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is in Jordan normal form.

The J is unique to A, but H is not. For example, $(2H)J(2H)^{-1} = A$ too. The above H was found using SymPy. What if we try to find H manually? (1) We start by finding a basis for $\ker(A - 3I_3)$ since $\lambda = 3$ is the only eigenvalue of A. (Check that $\det(A - \lambda I_3) = (3-\lambda)^3$!).

$$\ker\left(\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 5 & 0 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 4 & 0 & 0 \\ 1 & 5 & 0 \end{bmatrix}\right)$$ since the top row corresponds to $0x_1 + 0x_2 + 0x_3 = 0$, an equation true of all $\vec{x} \in \mathbb{R}^3$. This is an example where you must not simply delete the bottom row when finding a

basis of $\ker(A-\lambda I)$ because the bottom row $[1\ 5\ 0]$ is not a linear combination of $[0\ 0\ 0]$ & $[4\ 0\ 0]$. For a "random" matrix, dropping the bottom row works with high probability. This matrix, $\begin{bmatrix} 3 & 0 & 0 \\ 4 & 3 & 0 \\ 1 & 5 & 3 \end{bmatrix}$, was very carefully chosen to have repeating roots in $\det(A-\lambda I)$, and so is hardly random. The RREF of $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 5 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, which has kernel $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 1x_1 = 0 \ \&\ 1x_2 = 0 \right\}$, which has basis $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (since x_3 is free & $x_3 = 1$ is the simplest nonzero choice). Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which will be the first column of our "manual" H. The first column

of J will be $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ because $A\vec{v}_1 = 3\vec{v}_1$ (since $\vec{v}_1 \in \ker(A-3I_3)$) and $A = HJH^{-1}$ means J will be the matrix of the transformation $T\vec{x} = A\vec{x}$ in the basis $\vec{v}_1\ \vec{v}_2\ \vec{v}_3$ where $H = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, and $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ in column 1 then means $A\vec{v}_1 = 3\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3$.

Now we need $\vec{v}_2$ & $\vec{v}_3$...

② We solve $(A-3I_3)\vec{x}=\vec{v}_1$ for $\vec{x}$ to get $\vec{v}_2$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right] \xrightarrow[\text{RREF}]{} \begin{array}{c} \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/5 \\ \hline P & P & \text{free} & \end{array} \end{array} \iff \begin{array}{l} x_1 = 0 \\ x_2 = 1/5 \\ x_3 = x_3 \text{ (free)} \end{array}$$

As long as $\vec{v}_2$ is not in $\operatorname{span}\{\vec{v}_1\}$, any choice of x_3 will work. (We need $\vec{v}_1, \vec{v}_2, \vec{v}_3$ to be a basis, so we need $\vec{0} \neq \vec{v}_1$, $\vec{v}_2 \notin \operatorname{span}\{\vec{v}_1\}$, and $\vec{v}_3 \notin \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.)

$0 = x_3$ is the simplest choice (and any $x_3 \in \mathbb{R}$ would work).

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1/5 \\ 0 \end{bmatrix}$. Since $(A-3I_3)\vec{v}_2 = \vec{v}_1$, we have $A\vec{v}_2 = \vec{v}_1 + 3\vec{v}_2 + 0\vec{v}_3$, so the 2nd column of J is $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$.

(3) We solve $(A-3I_3)\vec{x} = \vec{v}_2$ to get $\vec{v}_3$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1/5 \\ 1 & 5 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1/20 \\ 0 & 1 & 0 & -1/100 \end{array}\right] \\ \begin{array}{ccc} P & P & \text{free} \end{array} \end{array} \iff \begin{array}{l} x_1 = 1/20 \\ x_2 = -1/100 \\ x_3 = x_3 \text{ (free)} \end{array}$$

For any choice of x_3, we will have $\vec{x} \notin \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$

(since $c_1 v_1 + c_2 v_2 = \begin{bmatrix} 0 \\ c_2/5 \\ c_1 \end{bmatrix} \neq \begin{bmatrix} 1/20 \\ -1/100 \\ x_3 \end{bmatrix}$ since $0 \neq 1/20$)

so pick $x_3 = 0$ for simplicity. Let $\vec{v}_3 = \begin{bmatrix} 1/20 \\ -1/100 \\ 0 \end{bmatrix}$.

Since $(A-3I_3)\vec{v}_3 = \vec{v}_2$, we have $A\vec{v}_3 = 0\vec{v}_1 + 1\vec{v}_2 + 3\vec{v}_3$

the third column of J is $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

(4) Put it all together: $H = \begin{bmatrix} 0 & 0 & 0.05 \\ 0 & 0.2 & -0.01 \\ 1 & 0 & 0 \end{bmatrix}$

& $J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. (Check that $A = HJH^{-1}$.)