A supremum proof of the I.V.T.

<u>Thm.</u> Suppose $f : D_f \to \mathbb{R}$, $a < b$, $[a,b] \subset D_f$, and f is continuous on $[a,b]$. Then, given y between $f(a)$ & $f(b)$, there exists x between a & b such that $f(x) = y$.

<u>Pf.</u> ① If $y = f(a)$ or $y = f(b)$, then $x = a$ or $x = b$ works. So, suppose y is strictly between $f(a)$ and $f(b)$.

If we prove case $f(a) < y < f(b)$, then, case $f(a) > y > f(b)$ follows from applying the first-case proof to $-f$ and $-y$ in place of f & y (for if f is cts. on $[a,b]$, so is $-f$).

So, suppose $f(a) < y < f(b)$.

(2) Let $S = \{x \in [a,b] : \forall z \in [a,x]\ f(z) < y\}$.
If $a \le w < z \in S$, then $w \in S$ too, so, letting $c = \sup(S)$, which exists because $a \in S$ and $S \le b$, we have $[a,c) \subset S$ because $a \le w < c$ implies $S \not\le w$, which implies $\exists z\ a \le w < z \in S$, which implies $w \in S$. So, $\forall x \in [a,c)$ $f(x) < y$. Also, $a \le c \le b$*, so $c \in D_f$.

(3) If $y = f(c) + \varepsilon$ for some $\varepsilon > 0$, then $c \ne b$, so $f(x) \to f(c)$ as $x \to c^+$, so, for $\delta > 0$ small enough, we have $c + \delta \le b$ and $\forall x \in [c, c+\delta)$ $f(x) \underset{\varepsilon}{\approx} f(c)$, so $\forall x \in [c, c+\delta)$ $f(x) < f(c) + \varepsilon = y$, so $\forall x \in [a, c+\delta) = [a,c) \cup [c, c+\delta)$ $f(x) < y$, so $\forall x \in [a, c + \frac{\delta}{2}]$ $f(x) < y$,

so $c+\frac{\delta}{2}\in S$, which contradicts $\underbrace{S \leq c}$.

Therefore, $\forall \varepsilon>0 \quad y \neq f(c)+\varepsilon$.

Thus, $y \leq f(c)$. $\boxed{\text{Recall } c=\sup(S) \Rightarrow S \leq c.}$

(4) If $y=f(c)-\varepsilon$ for some $\varepsilon>0$, then $c \neq a$, so $f(x) \to f(c)$ as $x \to c^-$, so, for $\delta>0$ small enough, we have $a \leq c-\delta$ and $\forall x \in (c-\delta, c]\ f(x) \underset{\varepsilon}{\approx} f(c)$, so $\forall x \in (c-\delta, c]\ f(x) > f(c)-\varepsilon = y$, so $\forall x \in (c-\delta, c]\ x \notin S$, which contradicts $[a, c) \subset S$. Therefore, $\forall \varepsilon>0 \quad y \neq f(c)-\varepsilon$. Thus, $y \geq f(c)$.

(5) We have shown that, letting $x=c$, $a \leq x \leq b$ and $y \leq f(x) \leq y$. $\square$

* Recall $S \leq b$ & $c=\sup(S) \Rightarrow c \leq b$.