1.3.1 a) $\frac{\sqrt{n^2-1}}{n} = \sqrt{1-\frac{1}{n^2}}$.

$\{n^2\}$ is $\nearrow$ $\Rightarrow$ $\{\frac{1}{n^2}\}$ is $\searrow$ $\Rightarrow$ $\{1-\frac{1}{n^2}\}$ is $\nearrow$

$\Rightarrow \left\{\sqrt{1-1/n^2}\right\} = \left\{\frac{\sqrt{n^2-1}}{n}\right\}$ is $\nearrow$.

$\sqrt{1-\frac{1}{n^2}} < \sqrt{1-0} = 1$, so 1 is an upper bound

$n^2 \to \infty \Rightarrow 1/n^2 \to 0 \Rightarrow \frac{\sqrt{n^2-1}}{n} = \sqrt{1-\frac{1}{n^2}} \to \sqrt{1-0}$

Thus, $\frac{\sqrt{n^2-1}}{n} \to 1$.

Remark: "$\{n^2\}$ is $\nearrow$ $\Rightarrow$ $\{1/n^2\}$ is $\searrow$" is valid because $n^2 > 0$. Sequences that switch signs, like $-5, -3, -1, 1, 3, 5, 7, \ldots$, are trickier...

1.3.1 b)

$(2-\frac{1}{n})(2+\frac{1}{n}) = 4 - n^{-2} \ (= 4 - \frac{1}{n^2})$

$\{n^2\}$ is $\nearrow \Rightarrow \{n^{-2}\}$ is $\searrow \Rightarrow \{4-n^{-2}\}$ is $\nearrow$.

4 is an upper bound of $\{4-n^{-2}\}$, and the limit of $\{4-n^{-2}\}$.

1.3.1 c) $\sum_0^n \sin^2 k\pi = \sum_0^n 0^2 = 0$, so the sequence $\left\{\sum_0^n \sin^2 k\pi\right\}$ is trivially $\nearrow$, bounded above by 0, and has limit 0.

1.3. (d)

$$\sum_{0}^{n+1} \sin^2\left(\frac{k\pi}{2}\right) - \sum_{0}^{n} \sin^2\left(\frac{k\pi}{2}\right) = \sin^2(n\pi/2) \geq 0,$$

so $\left\{ \sum_{0}^{n} \sin^2\left(\frac{k\pi}{2}\right) \right\}$ is $\nearrow$.

$$\sum_{0}^{n} \sin^2\left(\frac{k\pi}{2}\right) = \underbrace{0^2 + 1^2 + 0^2 + (-1)^2 + 0^2 + 1^2 + 0^2 + (-1)^2 + \cdots}_{n+1 \text{ terms}}$$

Therefore, $\sum_{0}^{n} \sin^2\left(\frac{k\pi}{2}\right) \geq \frac{n}{2}$.

Therefore, $\left\{ \sum_{0}^{n} \sin^2\left(\frac{k\pi}{2}\right) \right\}$ has no upper bound.

1.5.3) For $x \in [n, n+1]$, $\frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{n}}$,

so $\frac{1}{\sqrt{n}} = \int_n^{n+1} \frac{dx}{\sqrt{n}} \geq \int_n^{n+1} \frac{dx}{\sqrt{x}} = (\sqrt{n+1} - \sqrt{n}) \cdot 2.$

Therefore, $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$

$\geq 2\left[(\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{n+1} - \sqrt{n})\right]$

$= 2(\sqrt{n+1} - 1) \to \infty.$ Therefore, $\left\{\sum_1^n \frac{1}{\sqrt{k}}\right\}$ has no upper bound. (Alternative technique:

$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}} \Rightarrow \frac{1}{\sqrt{n}} \geq 2(\sqrt{n+1} - \sqrt{n}).$)

1.6.2) The sequence $\{a_n\} = \{n/2^n\}$ $(n \geq 1)$ is positive, so $a_{n+1} \leq a_n \iff \frac{a_{n+1}}{a_n} \leq 1$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{2^n(n+1)}{2^{n+1}n} = 2^{-1} \cdot \frac{n+1}{n} = 2^{-1}\left(1+\frac{1}{n}\right)$$

$$= \frac{1}{2}\left(1+\frac{1}{n}\right) \underset{\substack{\Uparrow \\ n \leq 1}}{\leq} \frac{1}{2}\left(1+\frac{1}{1}\right) = 1.$$

Thus, $\{a_n\}$ is $\searrow$, and, hence, monotone.

2.2.1 a)

$$-1 \le \cos n \le 1 \Rightarrow 2 \le 3 + \cos n \le 4$$

$$\Rightarrow \frac{1}{2} \ge \frac{1}{3+\cos n} \ge \frac{1}{4}$$

2.2.1 b) $|\sin n| \le 1$ & $1+n^2 \ge 1 \Rightarrow \frac{|\sin n|}{1+n^2} \le \frac{1}{1}$

$$\Leftrightarrow \left|\frac{\sin n}{1+n^2}\right| \le 1 \Leftrightarrow -1 \le \frac{\sin n}{1+n^2} \le 1.$$

The upper bound 1 is not sharp: If $n=0$, then $\frac{\sin n}{1+n^2} = \frac{\sin 0}{1+0^2} = 0$. If $n=1$, $\frac{\sin n}{1+n^2} = \frac{\sin(1)}{2} < 1$.

If $n \ge 2$, $\frac{\sin n}{1+n^2} \le \frac{1}{1+2^2} = \frac{1}{5} < \frac{\sin(1)}{2} < 1$.

Thus, $\frac{\sin(1)}{2}$ is the sharp upper bound.

2.4.3) $\left|\cos na + \frac{\sin nb}{n}\right| \le |\cos na| + \left|\frac{\sin nb}{n}\right|$

$= |\cos na| + \frac{|\sin nb|}{n} \le 1 + \frac{1}{n}.$

To show that ~~these~~ this bound~~s~~ ~~are~~ is attained, it is enough to find, given n, a & b such that $\cos na = 1 = \sin nb$. $a = \frac{2\pi}{n}$ & $b = \frac{\pi}{2n}$ work.

2.6.2) Given $a > 1$, we will find N such that $(n+1)/a^{n+1} \leq n/a^n$ for all $n \geq N$.

$$\frac{n+1}{a^{n+1}} \leq \frac{n}{a^n} \iff n+1 \leq na \quad \text{(multiplied both sides of } \leq \text{ by } a^{n+1}\text{)}$$

$$\iff 1 \leq n(a-1) \quad \text{(added } -n \text{ to both sides)}$$

$$\iff \frac{1}{a-1} \leq n. \quad \left(\text{multiplied by } \frac{1}{a-1}\right).$$

(Note: $a^{n+1} > 1^{n+1} > 0$ & $a-1 > 1-1 = 0$, so the above multiplications preserved the inequality $\leq$.)

$N = \frac{1}{a-1}$ works. $\square$